# Stability & Linear Classif$^n$.

Eq$^m$ sol$^n$ $\equiv$ fixed point (phase plane) $\Big\}$ when $\vec{x}\,' = \vec{0}$

## Stability of eq$^m$. sol$^n$'s.

# Phase plane Stability analysis

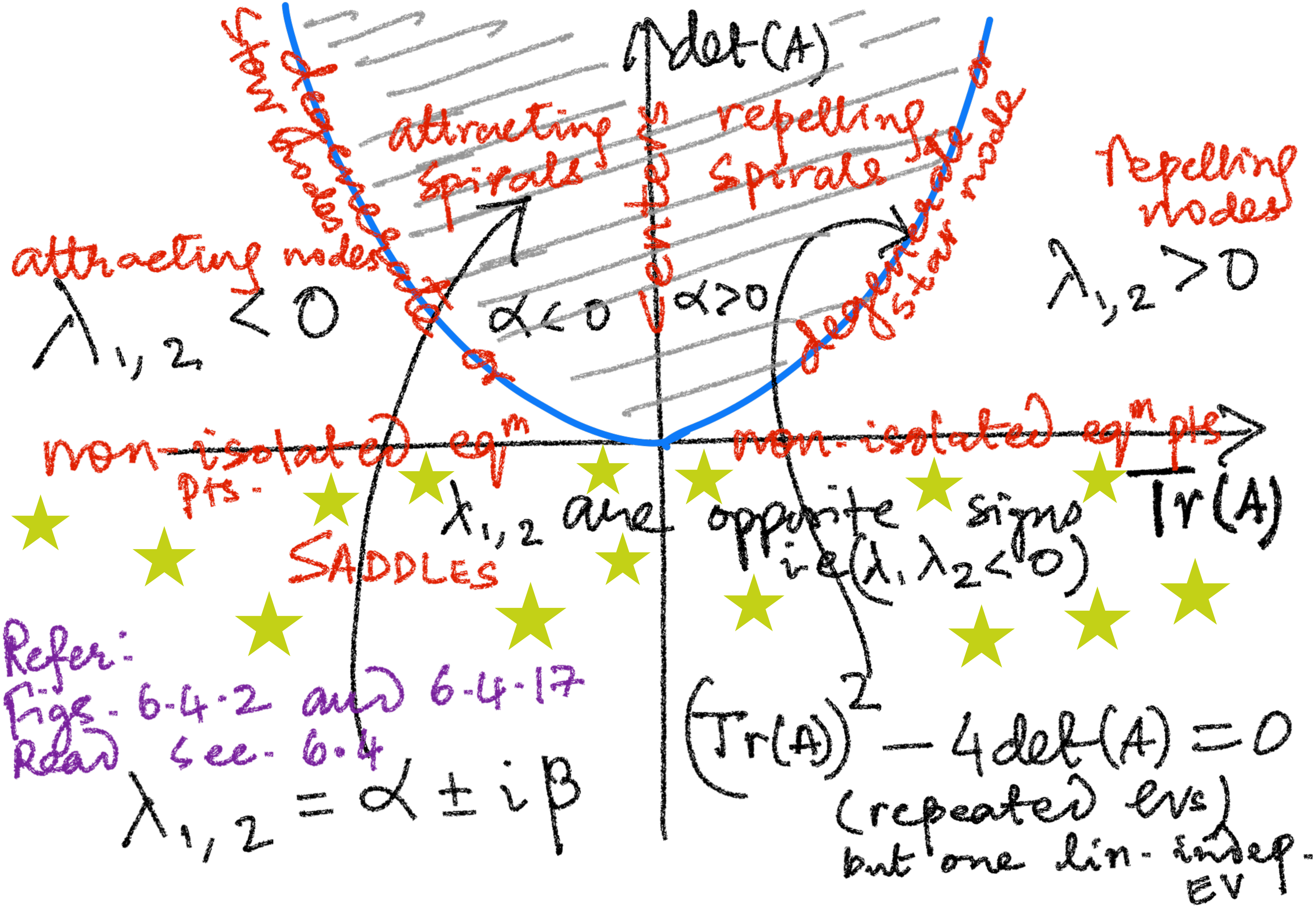
We will consider a $2 \times 2$ system of ODE for our analysis

$$\vec{x}' = A_{2\times 2}\vec{x}$$

$$|A - \lambda I| = \lambda^2 - Tr(A)\lambda + det(A) = 0$$

$$\text{So } \lambda_{1,2} = \frac{Tr(A) \pm \sqrt{(Tr(A))^2 - 4det(A)}}{2}$$

this expression determines whether we have stable or unstable solns.


det(A)
Tr(A)
attracting spirals
repelling spirals
centers
α<0
α>0
attracting nodes
λ1,2 < 0
repelling nodes
λ1,2 > 0
degenerate & star nodes
degenerate & star nodes
non-isolated eq^m pts.
non-isolated eq^m pts
λ1,2 are opposite signs i.e (λ1 λ2 < 0)
SADDLES
Refer: Figs. 6.4.2 and 6.4.17
Read Sec. 6.4
λ1,2 = α ± iβ
(Tr(A))^2 − 4det(A) = 0
(repeated evs)
but one lin. indep. EV

# Non-homogeneous Linear Systems

$$\vec{x}\,' = A(t)\,\vec{x} + \vec{f}(t)$$

So we should expect soln:

$$\vec{x} = \vec{x}_h + \vec{x}_p$$

## Case (1): Constant forcing $\vec{f}(t) = \vec{b}$

So why not try $\vec{x}_p$ = constant whence

$\vec{x}_p\,' = \vec{0} \Rightarrow A\vec{x}_p + \vec{b} = \vec{0} \Rightarrow \vec{x}_p = -A^{-1}\vec{b}$

$\therefore \quad \vec{x} = \vec{x}_h + \vec{x}_p = \vec{x}_h - A^{-1}\vec{b}$

eg ① Consider $\vec{x}' = A\vec{x} + \vec{b}$

$$= \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

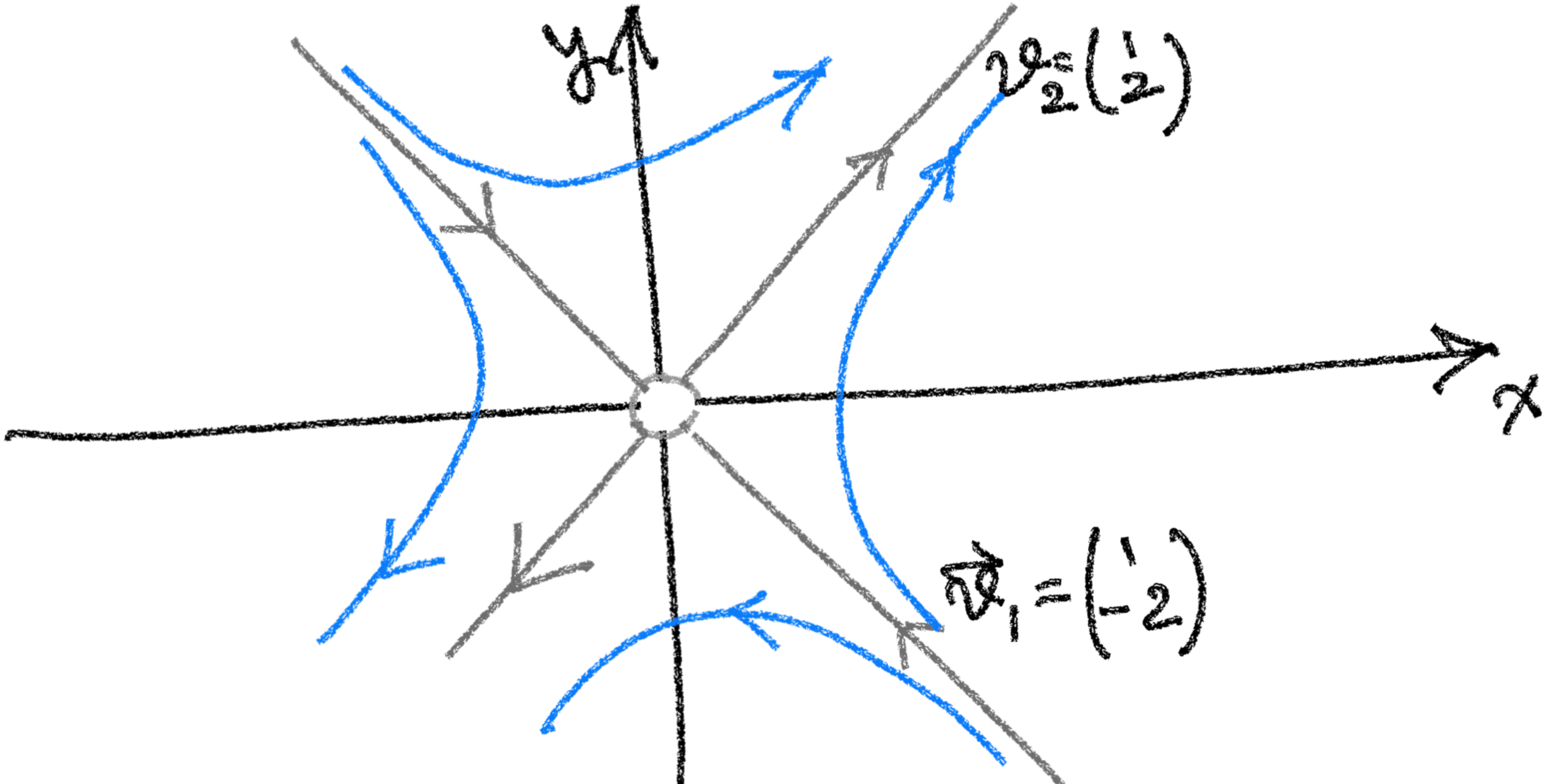
$$\vec{x}_h = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Now to find $\vec{x}_p$ :

$$A^{-1} = \frac{1}{3}\begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix} \Rightarrow \vec{x}_p = -A^{-1}\begin{pmatrix} 3 \\ 9 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$\therefore$ Gen. solⁿ :-

$$\vec{x}(t) = \underbrace{c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}}_{\vec{x}_h} + \underbrace{\begin{pmatrix} -2 \\ -1 \end{pmatrix}}_{\vec{x}_p}$$




phase portrait of the homogeneous system

What will the phase portrait of the non-homogeneous linear ODE w/ $\vec{b} = \begin{pmatrix} 3 \\ 9 \end{pmatrix}$ look like?

eg(2) $\vec{x}' = A\vec{x} + \vec{f}(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{x} + \begin{pmatrix} t-2 \\ 4t-1 \end{pmatrix}$

<u>Soln</u>:- $\vec{f}(t)$ is constituted of polynomials.

$$\therefore \quad \vec{x}_p = \begin{pmatrix} at+b \\ ct+d \end{pmatrix} \Rightarrow \vec{x}_p' = \begin{pmatrix} a \\ c \end{pmatrix}$$

Plug in

Plug in

We get $\begin{pmatrix} a \\ c \end{pmatrix} = t\begin{pmatrix} a+c+1 \\ 4a+c+4 \end{pmatrix} + \begin{pmatrix} b+d-2 \\ 4b+d-1 \end{pmatrix}$

Now compare coeffs. of $t$ and $t^0$:

$$a = b+d-2, \qquad a+c+1=0$$
$$c = 4b+d-1, \qquad 4a+c+4=0.$$

So we can now write a system of 4 linear eqns. in a, b, c, d :-

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & -1 & -2 \\ 1 & 0 & 1 & 0 & -1 \\ 0 & -4 & 1 & -1 & -1 \\ 4 & 0 & 1 & 0 & -4 \end{array}\right)$$

RREFied

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{aligned} a &= -1 \\ b &= 0 \\ c &= 0 \\ d &= 1 \end{aligned}$$

w/ these values:

$$\vec{x}_p = \begin{pmatrix} -t \\ 1 \end{pmatrix}$$

$\therefore$ Full soln.

$$\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} -t \\ 1 \end{pmatrix}$$

Phase portrait will be hard to draw by hand! Try using "quiver" in MATLAB.

** Due to a shortened & disruptive semester, we will ***not*** cover the following :-

(i) Variation of parameters to solve $y' + a(t)y = b(t)$ & its higher dim. avatars !

(ii) Matrix exponentials & Decoupling methods to solve coupled ODE using $D = P^{-1}AP$

(III) Introdn. to chaos dny
& phase portraits.

(IV) SVD

For (I) - (III) : read the Farlow book
For (IV) : read the book by Otto Bretscher

"NOT" included in EXAM.