# Lecture No. (6)

DATE 05/02/25

Initiation to a very important vector space :- Col$^m$ space of a matrix, Col(A).

Let's begin by two simple examples.

eg(1)

Consider the system of eqns :

$$\left.\begin{aligned} x + 2y \quad &= 1 \\ y + z &= 2 \\ 2x + 5y + z &= 3 \end{aligned}\right\} \text{———} (1)$$

If we employ the method of substitution & replace $x = 1-2y$ and $y = (2-z)$ in the third eq$^n$

$$2x + 5y + z = 3$$

i.e. $2(1-2y) + 5(2-z) + z = 3$

i.e. $2 - 4(2-z) + 10 - 5z + z = 3$

i.e. $-6 + \cancel{4z} + 10 - \cancel{4z} = 3$

i.e. $4 = 3$

this is clearly a contradiction

$\Rightarrow$ there is no solution to the system of eqs. (1).

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Had we written the system (1)

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$A \quad \vec{x} = \vec{b}$$

in terms of the col$^m$s of $A = \begin{pmatrix} | & | & | \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 \\ | & | & | \end{pmatrix}$

i.e. $x\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} + z\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$;

We notice that

$$2\bar{v}_1 + \bar{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \bar{v}_2$$

i.e. the col$^m$s. of $A$ do not form a linearly independent set of vectors (for the space spanned by the col$^m$s.);

And this explains the absence of a unique solution of (1). Infact, there is NO solution

If we check the determinant of this matrix $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 2 & 5 & 1 \end{pmatrix}$

$\det(A) = 0 \Rightarrow A^{-1}$ does not exist.

** For the inverse of the matrix $A$ to exist; the col$^m$s. of the matrix $A$ must constitute a linearly independent set of vectors.

** Later in the course we will see that when $A^{-1}$ DNE ; i.e. does not Exist.

when the col$^m$s. of $A$ are not linearly independent;

we have a rank deficient matrix.

eg (2) :-

Consider the system

$$\overset{A}{\begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix}} \overset{\vec{x}}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}} = \overset{\vec{b}}{\begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}}$$

$$\bar{v}_1 \quad \bar{v}_2 \quad \bar{v}_3 \quad \bar{v}_4$$

It's easy to check that $\bar{v}_2 = 3\bar{v}_1$

and $\bar{v}_4 = \frac{1}{3}\bar{v}_3 + \bar{v}_1$

Again we donot have a matrix w/ linearly independent col$^m$ vectors

$\Rightarrow$ We do <u>NOT</u> have a unique sol$^n$.

** Since $A \in M_{3\times 4}$ ; $A$ is NOT a square matrix

it's not possible to compute $\det(A)$ or $A^{-1}$.

But for eg(2); it's possible to find an <u>infinite family of solutions</u>, unlike the case in eg(1), where <u>NO</u> solutions exist.

Q) How to derive the infinitely many solutions?

this we will study in the next lectures.

** While we do NOT have a linearly independent set of col$^m$ vectors; and we do <u>not</u> have a unique solution; we do have a vector space $\equiv$ Col(A).

So we must obtain a "basis" of this vector space Col(A) and be able to <u>determine</u> it's dimension.

How? ← Row transformation of a Matrix
← Reduced Row Echelon form.

But that's the agenda for the next lecture.

Today we want to dwell a little more on the row & col$^m$ picture & use the algebraic & geometric framework of $A\vec{x} = \bar{b}$ & attempt to relate this w/ a physical intuition of a suitable model.

Model??

Spring-Mass System!!

Steady state $\Rightarrow \cancel{\frac{dx}{dt}} \to 0$.

& we have a linear model of such a system

$$A\bar{u} = \bar{f}.$$

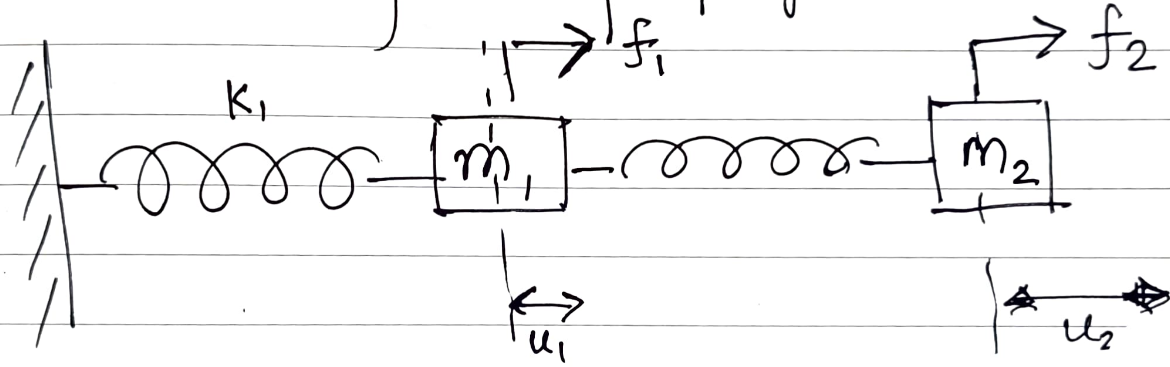
We will arrive at three important cases (Consequences).

(1) $\mathbb{A}\bar{u}=\bar{f}$ admits a unique solⁿ.

(2) $\mathbb{A}\bar{u}=\bar{f}$ admits $\infty$'ly many solⁿs

and, (3) $\mathbb{A}\bar{u}=\bar{f}$ has <u>NO</u> solⁿ.

Consider the following physical model:




WLOG, $u_2 > u_1$

→ $f_1$ and $f_2$ are steady (constant) forces

→ Restoring force is an opposing force

→ We want to find the values of $u_1$ and $u_2$ for the system will be in eqᵐ.

Sum of all forces on mass $m_1 = 0$

$$f_1 - k_1 u_1 + \underbrace{k_2(u_2 - u_1)}_{-k_2(u_1 - u_2)} = 0 \quad \text{— (1)}$$

Sum of all forces on mass $m_2 = 0$

$$f_2 - k_2(u_2 - u_1) = 0.$$

in $\mathbb{A}\bar{x} = \bar{b}$ form

$$\begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

$$\mathbb{A} \equiv \mathbb{K}$$

$$\mathbb{A}\bar{u} = \bar{f}$$

Case (1):-

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$k_1, k_2 = 1$ 
$f_1, f_2 = 1$

+ve dir$^n$ is the dir$^n$ away from the wall

$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \xrightarrow[(+)]{}$$

$$\underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{\bar{v}_1} \quad \underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\bar{v}_2}$$

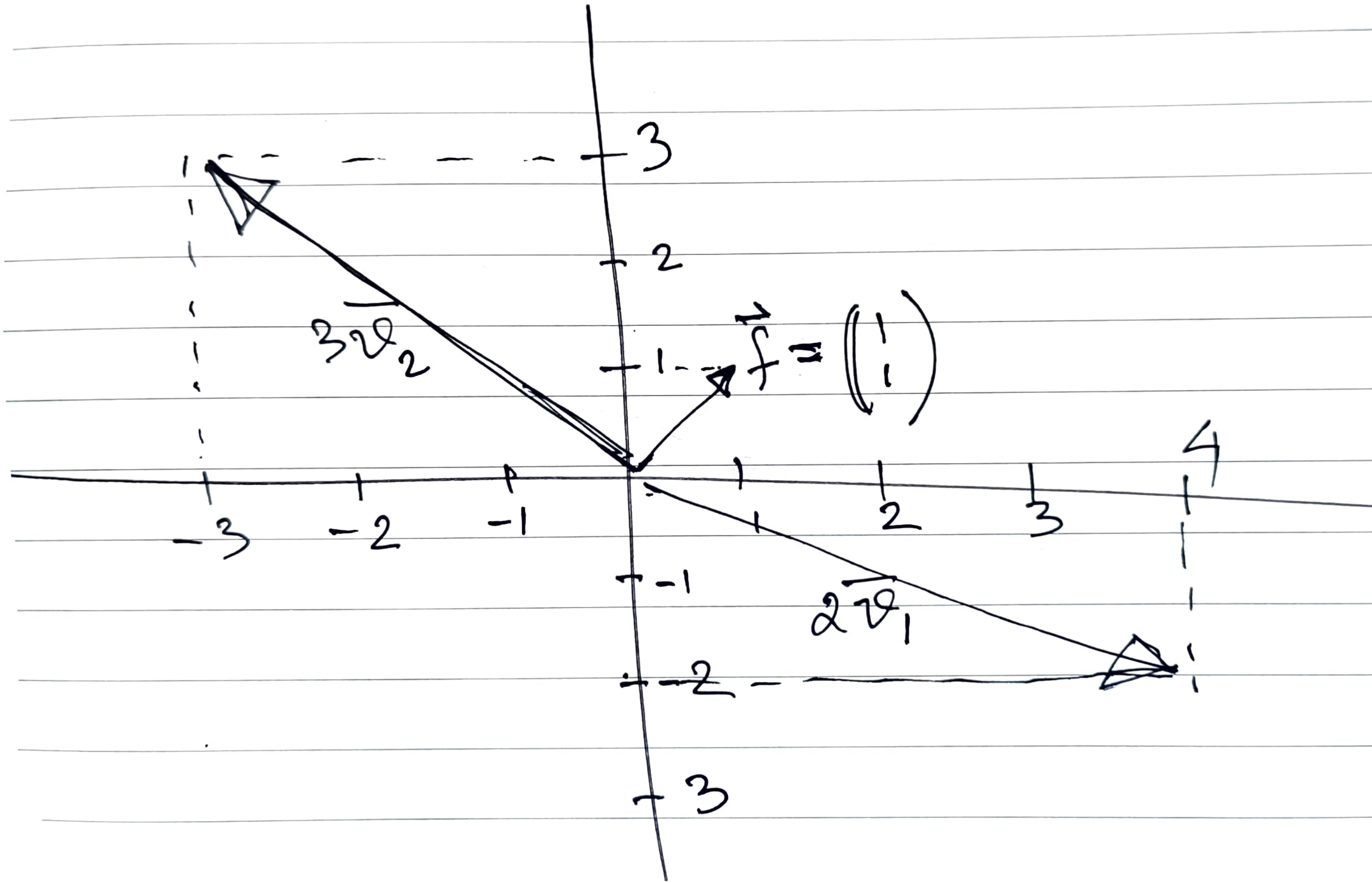
$\bar{v}_1$ and $\bar{v}_2$ are linearly independent Col$^m$ vectors

So we have a unique sol$^n$ $\begin{cases} u_1 = 2 \\ u_2 = 3 \end{cases}$

& the forcing f$^n$ $\bar{f}$ is in the span of the Col(A) b/c

$$2\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \bar{f}$$

$$2\bar{v}_1 + 3\bar{v}_2 = \bar{f}$$



3
2
1
-1
-2
3
-3
-2
-1
1
2
3
4
$3\vec{v}_2$
$\vec{f} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$2\vec{v}_1$

~~Case (iii)~~ :

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$K_1 = 0$

$K_2 = 1$

$\bar{f} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$\bar{v}_1$ $\quad$ $v_2$

Not linearly independent col^m vectors.

b/c $\bar{v}_2 = -\bar{v}_1$

Let's consider the linear comb^n. of the col^m vectors

$$u_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + u_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= u_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} - u_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= (u_1 - u_2) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \lambda \bar{v}_1$$

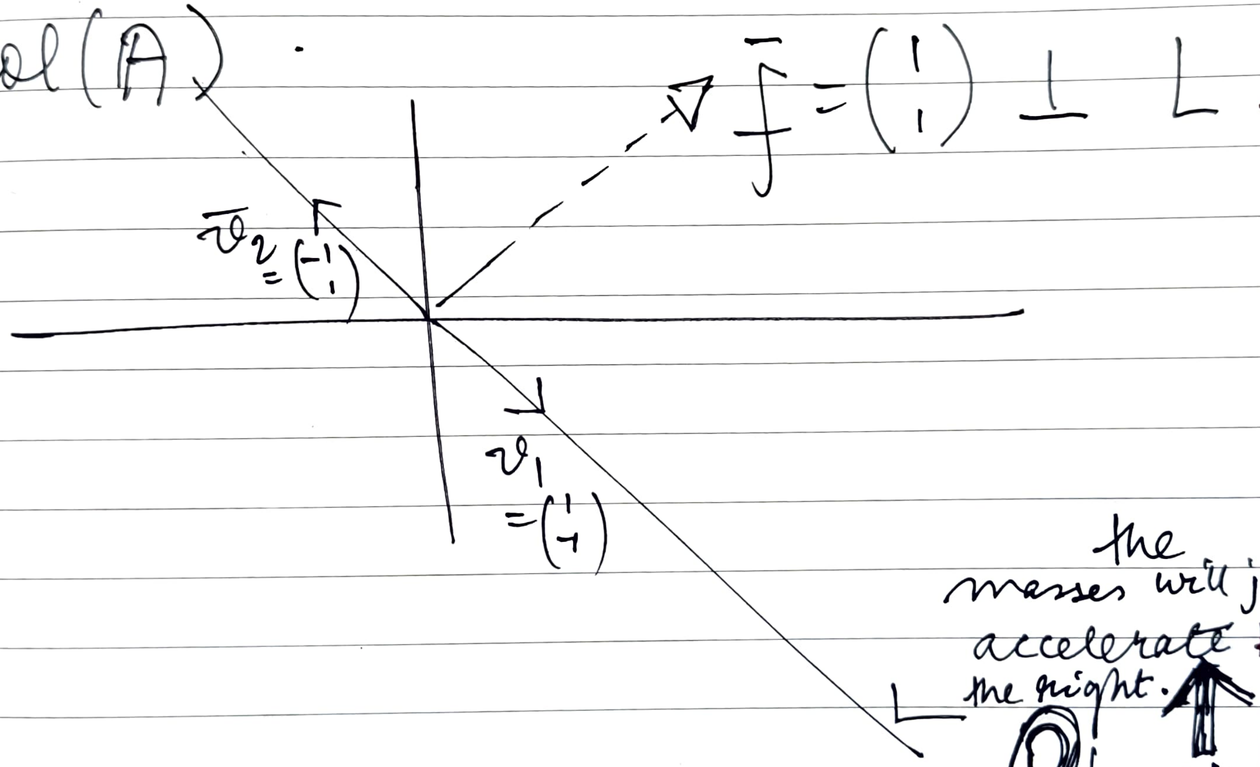
$u_1, u_2$ are just numbers.

$(u_1 - u_2) = \lambda$

So a linear combⁿ of these

colᵐ vectors just give us another vector that is either $\parallel^{\text{cel}}$ or anti-parallel to $\vec{v}_1$ OR along the line L but $\vec{f} = \begin{pmatrix}1\\1\end{pmatrix}$ is <u>not</u> along L (but $\perp$ to L).

So $\bar{f}$ is <u>not</u> in the <u>span</u> of Col(A).




the masses will just accelerate to the right.

Physically

$K_1 = 0$ means $m_1$ is no longer connected to the wall

algebraically

$$\begin{aligned} u_1 - u_2 &= 1 \\ -u_1 + u_2 &= 1 \\ \hline 0 &= 2 \end{aligned}$$

No eqᵐ solⁿ.

## Case (ii)

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$K_1 = 0$

$K_2 = 1$

$f_1 = 1$

$f_2 = -1$

$$\mathbb{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$\bar{v}_1$ $\bar{v}_2$ linearly dependent col$^n$ vectors

Now, even though the col($\mathbb{A}$) is <u>not</u> a linearly independent set

$\bar{f} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (u_1 - u_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is in the span of the col$^m$ space of $\mathbb{A}$.

Set $u_1 = \alpha$; $u_2 = \alpha + 1$ will satisfy $\bar{f} = (u_1 - u_2)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for any choice of $\alpha$

$\Rightarrow$ We have $\infty^{ly}$ many sol$^n$s. $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

(No unique sol$^n$) #.