Tutorial Worksheet-3 (WL3.2, WL4.1)
Direct computational method: Gauss elimination method, inspection of consistency of system of equations, Define vector norm and its properties, types of norms, define matrix norm, properties of matrix norms, spectral radius of matrix, Matrix decomposition, Iterative schemes to solve system of linear equations: Jacobi iterative method

Name and section: ______________________________

Instructor's name: ______________________________

1. Find the basis of column space and null space of below matrices.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

Solution: The RREF of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with first column as the only pivot column. Then the first column of the original matrix A, that is, $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, basis of column space matrix A.

Now let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in Null(A)$ then

$$x_1 + 2x_2 + 3x_3 = 0$$

here x_2 and x_3 are free variable.

so the basis of null space is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

The RREF of matrix $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ with first and second columns as the only pivot column. Then the first and second columns of the original matrix A, that is, $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is basis of column space of matrix A.

Now let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in Null(A)$ then

$$x_1 - x_3 - 2x_4 = 0$$

$$x_2 + 2x_3 + 3x_4 = 0$$

here x_3 and x_4 are free variable.

Let $x_3 = \alpha$ and $x_4 = \beta$, then $x_1 = \alpha + 2\beta$ and $x_2 = -2\alpha - 3\beta$. Then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \alpha + 2\beta \\ -2\alpha - 3\beta \\ \alpha \\ \beta \end{bmatrix} =$ $\alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \end{bmatrix}$. So the basis of null space is: $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2. Use Gauss-elimination and Gauss-Jordan-elimination to solve below linear system of equation(s).

$$\begin{aligned} 3x_1 + 5x_2 + 3x_3 &= 25 \\ 7x_1 + 9x_2 + 19x_3 &= 65 \\ -4x_1 + 5x_2 + 11x_3 &= 5 \end{aligned}$$

Solution: First we solve this system using Gauss-elimination method

The augmented matrix corresponding to the system of equations is:

$$\left(\begin{array}{ccc|c} 3 & 5 & 3 & 25 \\ 7 & 9 & 19 & 65 \\ -4 & 5 & 11 & 5 \end{array}\right)$$

$R_1 \to R_1/3$

$$\left(\begin{array}{ccc|c} 1 & \frac{5}{3} & 1 & \frac{25}{3} \\ 7 & 9 & 19 & 65 \\ -4 & 5 & 11 & 5 \end{array}\right)$$

$R_2 \to R_2 - 7R_1, R_3 \to R_3 + 4R_1$

$$\left(\begin{array}{ccc|c} 1 & \frac{5}{3} & 1 & \frac{25}{3} \\ 0 & \frac{-8}{3} & 12 & \frac{20}{3} \\ 0 & \frac{35}{3} & 15 & \frac{115}{3} \end{array}\right)$$

$R_2 \to R_2/\frac{-8}{3}$

$$\left(\begin{array}{ccc|c} 1 & \frac{5}{3} & 1 & \frac{25}{3} \\ 0 & 1 & -4.5 & -2.5 \\ 0 & \frac{35}{3} & 15 & \frac{115}{3} \end{array}\right)$$

$R_3 \to R_3 - \frac{35}{3}R_2$

$$\left(\begin{array}{ccc|c} 1 & \frac{5}{3} & 1 & \frac{25}{3} \\ 0 & 1 & -4.5 & -2.5 \\ 0 & 0 & 67.5 & 67.5 \end{array}\right)$$

$R_3 \to R_3/67.5$

$$\left(\begin{array}{ccc|c} 1 & \frac{5}{3} & 1 & \frac{25}{3} \\ 0 & 1 & -4.5 & -2.5 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

here we get

$$\begin{aligned} x_1 + \frac{5}{3}x_2 + x_3 &= \frac{25}{3} \\ x_2 - 4.5x_3 &= -2.5 \\ x_3 &= 1 \end{aligned}$$

using the back substitution we get

$$\begin{aligned} x_2 - 4.5 \times 1 = -2.5 &\implies x_2 = 2 \\ x_1 + \frac{5}{3} \times 2 + 1 = \frac{25}{3} &\implies x_1 = 4 \end{aligned}$$

This implies,

$$\begin{cases} x_1 = 4 \\ x_2 = 2 \\ x_3 = 1 \end{cases}$$

Now we solve this system using Gauss-Jordan-elimination method
The augmented matrix corresponding to the system of equations is:

$$\left(\begin{array}{ccc|c} 3 & 5 & 3 & 25 \\ 7 & 9 & 19 & 65 \\ -4 & 5 & 11 & 5 \end{array}\right)$$

$R_1 \to R_1/3$

$$\left(\begin{array}{ccc|c} 1 & \frac{5}{3} & 1 & \frac{25}{3} \\ 7 & 9 & 19 & 65 \\ -4 & 5 & 11 & 5 \end{array}\right)$$

$R_2 \to R_2 - 7R_1, R_3 \to R_3 + 4R_1$

$$\left(\begin{array}{ccc|c} 1 & \frac{5}{3} & 1 & \frac{25}{3} \\ 0 & \frac{8}{3} & 12 & \frac{20}{3} \\ 0 & \frac{35}{3} & 15 & \frac{115}{3} \end{array}\right)$$

$R_2 \to R_2/\frac{-8}{3}$

$$\left(\begin{array}{ccc|c} 1 & \frac{5}{3} & 1 & \frac{25}{3} \\ 0 & 1 & -4.5 & -2.5 \\ 0 & \frac{35}{3} & 15 & \frac{115}{3} \end{array}\right)$$

$R_1 \to R_1 - \frac{5}{3}R_2, R_3 \to R_3 - \frac{35}{3}R_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 8.5 & 12.5 \\ 0 & 1 & -4.5 & -2.5 \\ 0 & 0 & 67.5 & 67.5 \end{array}\right)$$

$R_3 \to R_3/67.5$

$$\left(\begin{array}{ccc|c} 1 & 0 & 8.5 & 12.5 \\ 0 & 1 & -4.5 & -2.5 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$R_1 \to R_1 - 8.5R_3, R_2 \to R_2 + 4.5R_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

This implies,

$$\begin{cases} x_1 = 4 \\ x_2 = 2 \\ x_3 = 1 \end{cases}$$

3. Find the matrix norms $||.||_\infty$ and $||.||_1$ of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Solution: let $A = (a_{(ij)})$ be a $m \times n$ matrix then

$$||A||_\infty = \max_{1 \le i \le n} \sum_{j=1}^{m} |a_{ij}|$$

so here

$$||A||_\infty = \max\{2, 4, 1\} = 4$$

$$||A||_1 = \max_{1 \le j \le m} \sum_{i=1}^{n} |a_{ij}|$$

so here

$$||A||_1 = \max\{3, 3, 1\} = 3$$

4. Solve the system of linear equation by Jacobi Method and Gauss Seidel Method up to 2th iteration with initial approximation as $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$

$$\begin{aligned} 5x - y + z &= 10 \\ 2x + 4y &= 12 \\ x + y + 5z &= -1 \end{aligned}$$

Solution: First we solve using Jocabi Method
Write the matrix representation of the system:

$$AX = b, \quad \text{where } A = \begin{bmatrix} 5 & -1 & 1 \\ 2 & 4 & 0 \\ 1 & 1 & 5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } b = \begin{bmatrix} 10 \\ 12 \\ -1 \end{bmatrix}$$

From A, find $D, L, U, L+U$ and therefore find $D^{-1}, D^{-1}b$ and $D^{-1}(L+U)$:

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, L+U = \begin{bmatrix} 0 & -1 & 1 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}, D^{-1}(L+U) = \begin{bmatrix} 0 & -1/5 & 1/5 \\ 2/4 & 0 & 0 \\ 1/5 & 1/5 & 0 \end{bmatrix}, D^{-1}b = \begin{bmatrix} 2 \\ 3 \\ -1/5 \end{bmatrix}$$

Consider the Jacobi iterative formula:

$$x^{(k)} = -D^{-1}(L+U)x^{(k-1)} + D^{-1}b$$

that is,

$$x^{(k)} = -\begin{bmatrix} 0 & -1/5 & 1/5 \\ 2/4 & 0 & 0 \\ 1/5 & 1/5 & 0 \end{bmatrix} x^{(k-1)} + \begin{bmatrix} 2 \\ 3 \\ -1/5 \end{bmatrix}$$

For $k = 1$,

$$x^{(1)} = -\begin{bmatrix} 0 & -1/5 & 1/5 \\ 2/4 & 0 & 0 \\ 1/5 & 1/5 & 0 \end{bmatrix} x^{(0)} + \begin{bmatrix} 2 \\ 3 \\ -1/5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1/5 \end{bmatrix}$$

For $k = 2$,

$$x^{(2)} = -\begin{bmatrix} 0 & -1/5 & 1/5 \\ 2/4 & 0 & 0 \\ 1/5 & 1/5 & 0 \end{bmatrix} x^{(1)} + \begin{bmatrix} 2 \\ 3 \\ -1/5 \end{bmatrix}$$

$$x^{(2)} = -\begin{bmatrix} 0 & -1/5 & 1/5 \\ 2/4 & 0 & 0 \\ 1/5 & 1/5 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1/5 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1/5 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} 16/25 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1/5 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} 66/25 \\ 2 \\ -6/5 \end{bmatrix}$$

Now we solve using Gauss Seidel Method

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, U = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, D + L = \begin{bmatrix} 5 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 1 & 5 \end{bmatrix}$$

$$(D+L)^{-1} = \begin{bmatrix} 1/5 & 0 & 0 \\ -1/10 & 1/4 & 0 \\ -1/50 & -1/20 & 1/5 \end{bmatrix}$$

$$(D+L)^{-1}U = \begin{bmatrix} 0 & -1/5 & 1/5 \\ 0 & 1/10 & -1/10 \\ 0 & 1/50 & -1/50 \end{bmatrix}, (D+L)^{-1}b = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

Consider the Gauss-Seidel iterative formula:

$$x^{(k)} = -(D+L)^{-1}Ux^{(k-1)} + (D+L)^{-1}b$$

For $k = 1$,

$$x^{(1)} = -\begin{bmatrix} 0 & -1/5 & 1/5 \\ 0 & 1/10 & -1/10 \\ 0 & 1/50 & -1/50 \end{bmatrix} x^{(0)} + \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

For $k = 2$,

$$x^{(2)} = -\begin{bmatrix} 0 & -1/5 & 1/5 \\ 0 & 1/10 & -1/10 \\ 0 & 1/50 & -1/50 \end{bmatrix} x^{(1)} + \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$x^{(2)} = -\begin{bmatrix} 0 & -1/5 & 1/5 \\ 0 & 1/10 & -1/10 \\ 0 & 1/50 & -1/50 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} 3/5 \\ -3/10 \\ -3/50 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 1.7 \\ -1.06 \end{bmatrix}$$