Module 3

Ordinary Differential Equations and their Solution

Engineering Mathematics in Action: FM112

Prof. Amrik Sen
Prof. Sanjay K. Bose

Linear Ordinary Differential Equations
(ODE)
with
Constant Coefficients

Form of Linear ODEs with Constant Coefficients

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots\ldots\ldots + a_1 \frac{dy}{dx} + a_0 y = 0$$

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots\ldots\ldots\ldots\ldots + a_1 y' + a_0 y = 0 \qquad y^{(j)} = \frac{d^j y}{dx^j}$$

$\mathcal{L}$: = $a_0 + a_1 \frac{d}{dx} + \ldots\ldots\ldots\ldots + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \frac{d^n}{dx^n}$ is the **linear differential operator**

$$\mathcal{L}[y(x)] = 0$$

Form of the Solution: We seek a solution of the form $y(x)=e^{rx}$

$$\mathscr{L}(e^{rx}) = \mathcal{P}(r)e^{rx}$$

where $$\mathcal{P}(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0$$

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Characteristic Equation

$$\mathcal{P}(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$$

Solution Set of the ODE depends on the nature of the roots of the Characteristic Equation:

$$\mathcal{P}(r) = r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0$$

1. $\mathcal{P}(r)$ has *n* distinct real roots $r_1, \dots \dots r_n$

2. The roots are all real but there are some multiple roots,

   e.g. *m* multiple roots (*m*≤*n*) for $r = r_0$ and the other (*n*-*m*) roots are distinct (*other combinations of multiple roots also possible*)

3. Complex Roots **For repeated complex roots, the approach for (2) is followed**

Case I - When $\mathcal{P}(r)$ has *n* distinct real roots $r_1, \ldots \ldots r_n$

Then $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + \ldots\ldots\ldots + c_n e^{r_n x}$

Example $y'' + 5y' + 6y = 0$

Characteristic Equation: $r^2 + 5r + 6 = 0 \Rightarrow (r+2)(r+3) = 0$

Characteristic Roots: $r_1 = -2,\ r_2 = -3$

$y(x) = c_1 e^{-2x} + c_2 e^{-3x}$ The set $\{e^{-2x}, e^{-3x}\}$ is a basis for the solution space $\mathbb{S}$, and dim$\mathbb{S}$=2

For an Initial Value Problem (IVP) with initial conditions $y(0) = 1, y'(0) = 0$, we get -

$$y(x) = 3e^{-2x} - 2e^{-3x}$$

Case II - The roots are all real but there are some multiple roots, e.g. *m* multiple roots ($m \le n$) for $r = r_0$ and the other (n-m) roots are distinct,

Then, $$y(x) = (c_1 + c_2 x + \ldots\ldots + c_m x^{m-1})e^{r_0 x} + d_1 e^{r_1 x} + d_2 e^{r_2 x} + \ldots\ldots\ldots + d_{n-m} e^{r_{(n-m)} x}$$

Similar approach when there are more than one such set of multiple roots of the characteristic equation

Example $y'' - 4y' + 4y = 0$

Characteristic Equation: $r^2 - 4r + 4 = 0 \;\Rightarrow\; (r-2)^2 = 0$

Characteristic Roots: Double Root at $r = 2$

$y(t) = c_1 e^{2x} + c_2 x e^{2x}$ The set $\{e^{2x}, xe^{2x}\}$ is a basis for the solution space $\mathbb{S}$, and dim$\mathbb{S}$=2

For an Initial Value Problem (IVP) with initial conditions $y(0) = 1, y'(0) = 1$, we get $c_1 = 1, c_2 = 1 - 2c_1 = -1$

$$y(x) = e^{2x} - xe^{2x}$$

Another Example for Case II

$$\frac{d^5y}{dt^5}+3\frac{d^4y}{dt^4}+3\frac{d^3y}{dt^3}+\frac{d^2y}{dt^2}=0$$

Characteristic Equation: $r^5+3r^4+3r^3+r^2=(r+1)^3r^2=0$

Characteristic Roots: $r=-1$ Triple Root

$r=0$ Double Root

For this, the form of the solution would be –

$$y(t)=(c_1+c_2t+c_3t^2)e^{-t}+c_4+c_5t$$

Case III - Complex (conjugate) Roots, possibly multiple roots along with real roots

The solution would be of the form $y = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x})$

$$+ \text{linear combination of real solutions}$$

For repeated complex roots, the approach for (2) is followed

Example $\frac{d^4 y}{dt^4} + 8\frac{d^2 y}{dt^2} + 16y = 0$

Characteristic Eq. $r^4 + 8r^2 + 16 = 0 \quad (r^2 + 4)^2 = 0$

Characteristic Roots: $r = \pm 2i$ **Double Roots**

The solution will be of the form –

$$y = (c_1 + c_2 t)cos2t + (c_3 + c_4 t)sin2t$$

Example: Solving ODE with Constant Coefficient

Problem: $\epsilon y'' + y = 0; \quad y(0) = 0, y(1) = 1$ where ϵ is a constant (for now)

Solution: This is an ODE with constant coefficients.

Characteristic Equation: $r^2 + \frac{1}{\epsilon} = 0$

Characteristic Roots: $r = \pm \frac{i}{\sqrt{\epsilon}}$

$$y(x) = c_1 e^{\frac{i}{\sqrt{\varepsilon}}x} + c_2 e^{\frac{-i}{\sqrt{\varepsilon}}x}$$

Applying the boundary conditions –

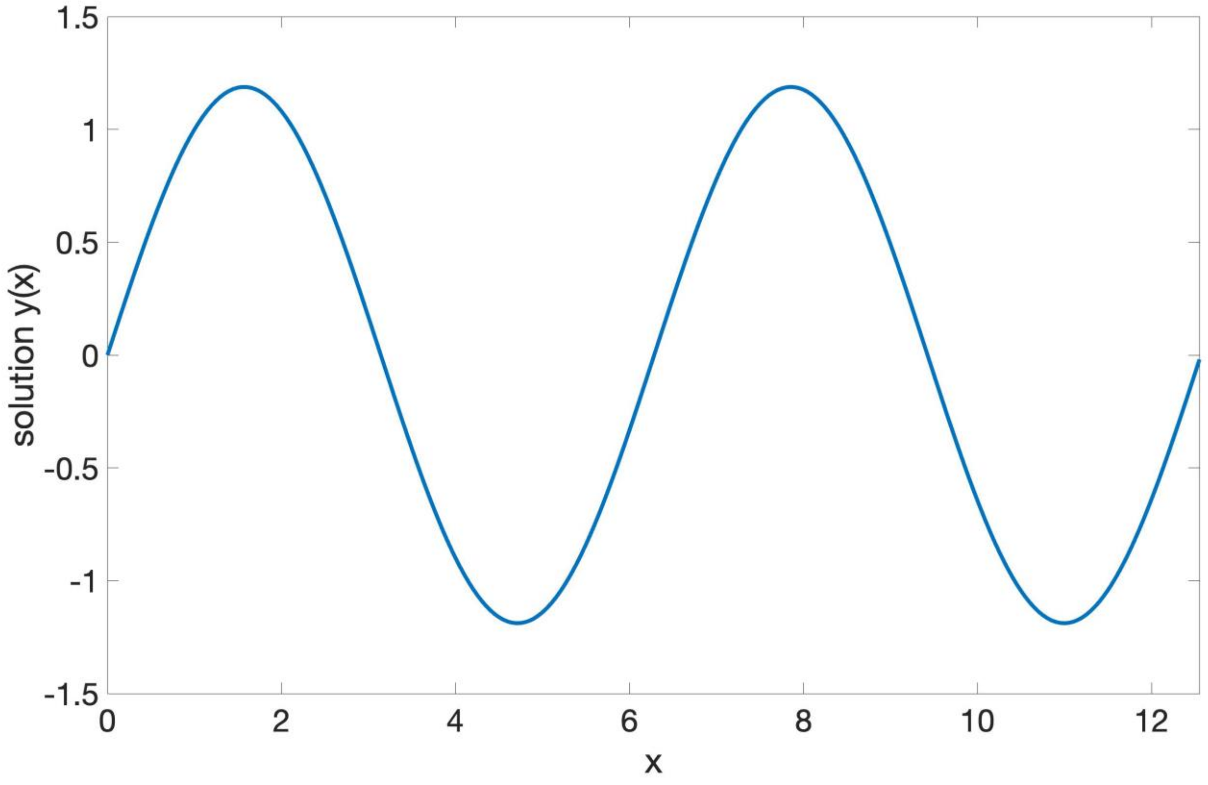
$$y(0) \Rightarrow c_1 = -c_2 = c \quad \& \quad y(1) = 1 \Rightarrow c = \frac{1}{2i\sin(\frac{1}{\sqrt{\epsilon}})}$$

We get,

$$y(x) = \frac{\sin\left(\frac{x}{\sqrt{\epsilon}}\right)}{\sin\left(\frac{1}{\sqrt{\epsilon}}\right)}$$ as the final solution

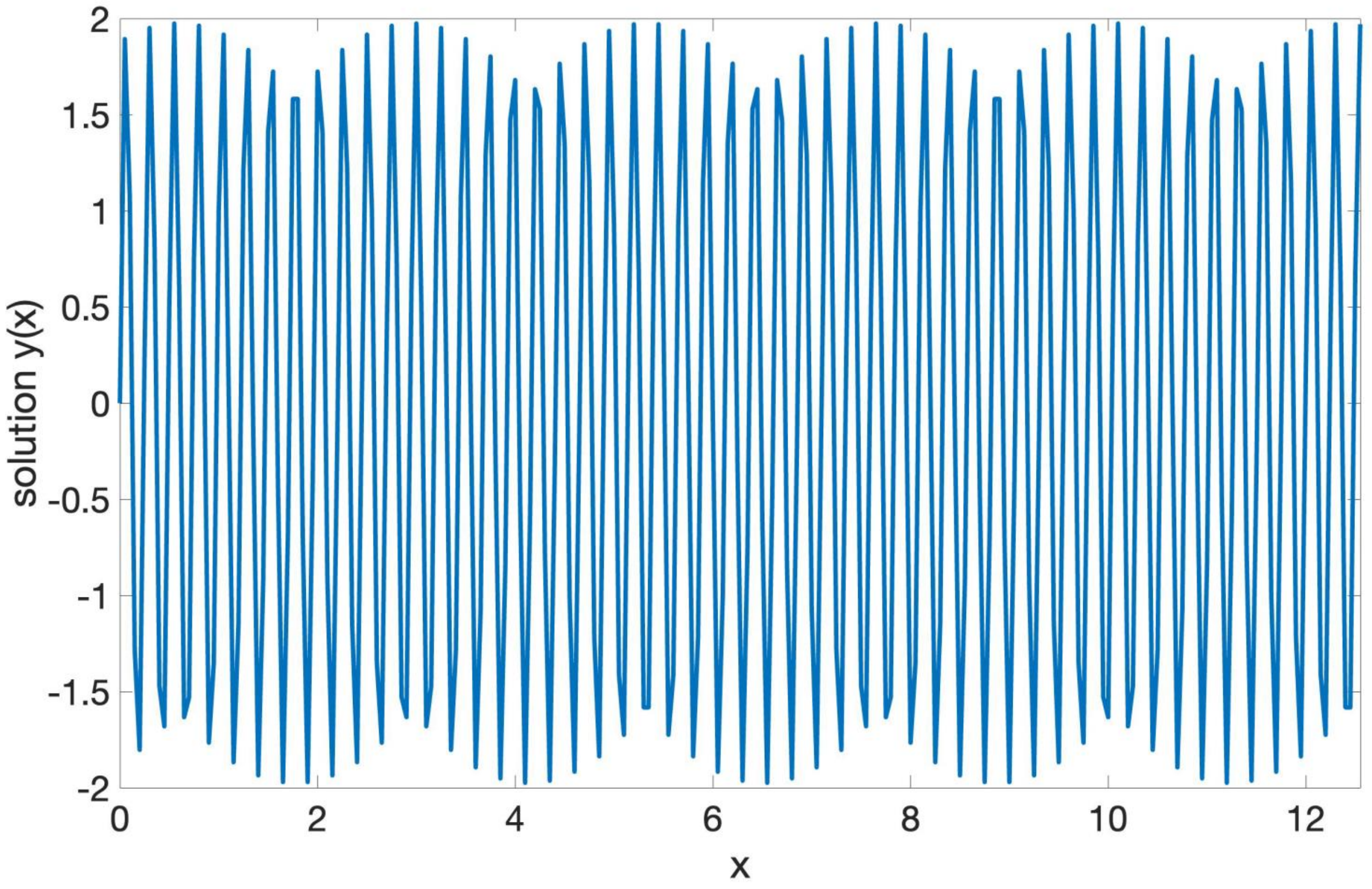
We would like to examine the behavior of $y(x) = \frac{\sin\left(\frac{x}{\sqrt{\epsilon}}\right)}{\sin\left(\frac{1}{\sqrt{\epsilon}}\right)}$ as $\epsilon \to 0^+$

Check: $\epsilon = 1$ gives $y \sim \sin x$



$\epsilon = 1.0$

We would like to examine the behavior of $y(x) = \frac{\sin\left(\frac{x}{\sqrt{\epsilon}}\right)}{\sin\left(\frac{1}{\sqrt{\epsilon}}\right)}$ as $\epsilon \to 0^+$



$\epsilon = 0.0001$

For $\epsilon = 0.0001$, we get rapid oscillations, as shown, which increase as ϵ decreases

Singular Perturbation Problems: (A prelude to advanced mathematics for later semesters)

Some options:

1. Since $\epsilon \to 0^+$, ignore terms comprising ϵ. Then the ODE $\epsilon y'' + y = 0$ becomes $y = 0$. Clearly, $y(1) = 1$ contradicts $y(x) = 0$. **BAD OPTION**

2. WKB Analysis (*Wentzel–Kramers–Brillouin*) seeks solutions of the form -

$$y(x) \sim \exp\left(\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)\right), \quad \delta \to 0$$

Using the above $y(x)$ in $\epsilon y'' + y = 0$, we obtain a hierarchy of closed differential equations for $S_n(x)$, solvable at every order of ϵ, to construct the asymptotic solution $y(x) \sim$