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Q) Apply the Gram-Schmidt orthonormal" process to find the ON bases of the vector spanned by the basis vectors.

$\bar{v}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$; $\bar{v}_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

Soln:- $\bar{u}_1 = \dfrac{\bar{v}_1}{\|\bar{v}_1\|} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hat{e}_1$

$$\bar{v}_2^{\parallel} = \text{proj}_{\bar{u}_1} \bar{v}_2 = \langle \bar{u}_1, \bar{v}_2 \rangle \bar{u}_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$$\bar{v}_2^{\perp} = \bar{v}_2 - \bar{v}_2^{\parallel} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\therefore \bar{u}_2 = \frac{\bar{v}_2^{\perp}}{\|\bar{v}_2^{\perp}\|} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hat{e}_2$$

$$\bar{v}_3^{\parallel} = \langle \bar{u}_1, \bar{v}_3 \rangle \bar{u}_1 + \langle \bar{u}_2, \bar{v}_3 \rangle \bar{u}_2$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$\bar{v}_3^{\perp} = \bar{v}_3 - \bar{v}_3^{\parallel} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore \bar{u}_3 = \frac{\bar{v}_3^{\perp}}{\|\bar{v}_3^{\perp}\|} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \hat{e}_3.$$

$\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ are the ON bases of R^3 spanned by $\bar{v}_1$, $\bar{v}_2$, and $\bar{v}_3$.

QR factorization

The Gram-Schmidt process represents a change of basis from the old basis $\vec{v}_1, \ldots, \vec{v}_m$ to a new orthonormal basis $\vec{u}_1, \ldots, \vec{u}_m$ of V. The QR factorization involves a change of basis matrix R such that

$$\begin{pmatrix} | & | & | & | & | \\ \vec{v}_1 & \cdot & \cdot & \cdot & \vec{v}_m \\ | & | & | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | & | & | \\ \vec{u}_1 & \cdot & \cdot & \cdot & \vec{u}_m \\ | & | & | & | & | \end{pmatrix} R$$

i.e. $M = QR$;

where R is an upper triangle matrix with entries:
$r_{11} = ||\vec{v}_1||$, $r_{jj} = ||\vec{v}_j^{\perp}||$ (for $j = 2,\ldots,m$), and $r_{ij} = \langle \vec{u}_i, \vec{v}_j \rangle$ (for $i < j$) .

Example: Find the QR factorization of the matrix $M = \begin{pmatrix} 2 & 2 \\ 1 & 7 \\ -2 & -8 \end{pmatrix}$.

Solution: $Q = \frac{1}{3}\begin{pmatrix} 2 & -2 \\ 1 & 2 \\ -2 & -1 \end{pmatrix}$ and $R = \begin{pmatrix} 3 & 9 \\ 0 & 6 \end{pmatrix}$.