Linear Algebra – Module 2

Engineering Mathematics In Action

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Matrix Factorizations and Applications

LU Factorization

($\boldsymbol{A} = \boldsymbol{LU}$ where ***A*** is an $n \times n$ *square matrix*)

Consider the following set of equations: ***Ax=b***

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots\ a_{1n}x_n &= b_1 \ldots\ldots\ldots\ldots (i) \\
a_{21}x_1 + a_{22}x_2 + \cdots\ a_{2n}x_n &= b_2 \ldots\ldots\ldots\ldots (ii) \\
\cdot\cdot \quad &= \cdot \\
\cdot\cdot \quad &= \cdot \\
\cdot\cdot \quad &= \cdot \\
a_{n1}x_1 + a_{n2}x_2 + \cdots\ a_{nn}x_n &= b_n \ldots\ldots\ldots\ldots (n^{th} eq.)
\end{aligned}
$$

with ***A*** (*n x n*), ***x*** (*n x 1*) and ***b*** (*n x 1*) defined as usual

LU Factorization Factorize ***A*** in ***Ax=b*** as ***A=LU***

Objective: Factorize ***A=LU*** where ***L*** is a matrix in the lower-triangular form and ***U*** is a matrix in the upper-triangular form as given below.

$$\mathbf{A} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ m_{21} & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ m_{n1} & m_{n2} & \cdot & \cdot & m_{n(n-1)} & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdot & \cdot & \cdot & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdot & \cdot & \cdot & a_{2n}^{(2)} \\ \cdot & 0 & \cdot & \cdot & & \cdot \\ \cdot & \cdot & 0 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & a_{(n-1)n}^{(n-1} \\ 0 & \cdot & \cdot & \cdot & 0 & a_{nn}^{(n} \end{pmatrix}}_{U}$$

It turns out that ***L*** is the matrix of the **multipliers** m_{jk} of the Gauss Elimination method with the main diagonal as 1, ……., 1 and with ***U*** as the matrix of the triangular system at the end of the Gauss Elimination. Details of how to do this are described later.

Example of LU Factorization

$$\begin{pmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\boldsymbol{A} \qquad\qquad \boldsymbol{L} \qquad\qquad \boldsymbol{U}$$

Why do *LU* Factorization?

Since ***Ax=b*** implies ***LUx=b***

or ***Ly=b*** with ***y=Ux***

Knowing the factors ***L*** and ***U***, we can first solve for ***y*** in ***Ly=b*** by forward substitution

We then solve for ***x*** in ***Ux***=***y*** by backward substitution

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ m_{21} & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ m_{n1} & m_{n2} & \cdot & \cdot & m_{n(n-1)} & 1 \end{pmatrix}}_{\boldsymbol{L}} \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \\ y_n \end{pmatrix}}_{\boldsymbol{y}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \\ b_n \end{pmatrix}}_{\boldsymbol{b}}$$

$$\underbrace{\begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdot & \cdot & \cdot & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdot & \cdot & \cdot & a_{2n}^{(2)} \\ \cdot & 0 & \cdot & \cdot & & \cdot \\ \cdot & \cdot & 0 & \cdot & & \cdot \\ \cdot & \cdot & \cdot & 0 & \cdot & a_{(n-1)n}^{(n-1} \\ 0 & \cdot & \cdot & \cdot & 0 & a_{nn}^{(n} \end{pmatrix}}_{\boldsymbol{U}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \\ x_n \end{pmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \\ y_n \end{pmatrix}}_{\boldsymbol{y}}$$

Use Forward Substitution to get ***y*** starting from Row 1

Use Backward Substitution to get ***x*** starting from Row n

Why do *LU* Factorization?   *Solving n equations for n unknowns*

- Gauss Elimination typically requires about $2n^3/3$ operations

- The ***LU*** decomposition requires $n^3/3$ operations while solving ***Lux=b*** requires only about n^2 operations

- Therefore the ***LU*** decomposition method generally requires fewer operations than the Gauss Elimination method.

- One more advantage of the ***LU*** decomposition method is that, given a fixed ***A*,** changing ***b*** can give us the different solutions much more efficiently than Gauss Elimination. (Why? … Because we need to do the *LU decomposition* only once!)

When can one do *LU* decomposition for a matrix *A*?

For a **non-singular** matrix, one can always **reorder** the rows to get a matrix for which one can do ***LU*** factorization.

The objective of the reordering is to make the diagonal coefficients non-zero. Once this is done, it would be possible to do the ***LU*** factorization.

The ***LU*** factorization is unique if we ensure that the diagonal term of the ***L*** matrix is all 1's

How to do *LU* decomposition

Reorder rows to ensure non-zero diagonal terms

$$
\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}}_{\boldsymbol{A}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ m_{21} & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ m_{n1} & m_{n2} & \cdot & \cdot & m_{n(n-1)} & 1 \end{pmatrix}}_{\boldsymbol{L}} \underbrace{\begin{pmatrix} u_{11} & u_{12} & \cdots & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & \cdots & u_{2n} \\ 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & u_{nn} \end{pmatrix}}_{\boldsymbol{U}}
$$

There are several approaches to find ***L*** and ***U***

LU Factorization Example

Explicitly solve for the u_{ij}s and the m_{ij}s

$$\begin{pmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -7 \\ 26 \end{pmatrix}$$

$$A = \left[a_{jk}\right] = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

$Solution: x_1 = 4, x_2 = -1; x_3 = \frac{1}{2}$ shown later

$$a_{11} = 3 = u_{11} \qquad a_{12} = 5 = u_{12} \qquad a_{13} = 2 = u_{13}$$

$$a_{21} = 0 = m_{21}u_{11} \qquad a_{22} = 8 = m_{21}u_{12} + u_{22} \qquad a_{23} = 2 = m_{21}u_{13} + u_{23}$$

$$m_{21} = 0 \qquad u_{22} = 8 \qquad u_{23} = 2$$

$$a_{31} = 6 = m_{31}u_{11} \qquad a_{32} = 2 = m_{31}u_{12} + m_{32}u_{22} \qquad a_{33} = m_{31}u_{13} + m_{32}u_{23} + u_{33}$$

$$= m_{31}(3) \qquad = 2(5) + m_{32}(8) \qquad = 2(2) + -1(2) + u_{33}$$

$$m_{31} = 2 \qquad m_{32} = -1 \qquad u_{33} = 6$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\qquad L \qquad\qquad\qquad U$$

Reorder rows to ensure non-zero diagonal terms

$$
\boldsymbol{A} \quad \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot & 0 \\ m_{21} & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & & & 0 \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & & \cdot & \cdot \\ m_{n1} & m_{n2} & \cdot & \cdot & m_{n(n-1)} & 1 \end{pmatrix}}_{\boldsymbol{L}} \underbrace{\begin{pmatrix} u_{11} & u_{12} & \dots & \dots & u_{1n} \\ 0 & u_{22} & \dots & \dots & u_{2n} \\ 0 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & u_{nn} \end{pmatrix}}_{\boldsymbol{U}}
$$

$$u_{1k} = a_{1k} \qquad k = 1,\dots\dots,n$$

$$m_{j1} = \frac{a_{j1}}{u_{11}} \qquad j = 2,\dots\dots,n$$

$$u_{jk} = a_{jk} - \sum_{s=1}^{j-1} m_{js} u_{sk} \qquad k = j,\dots\dots,n;\ j \geq 2$$

$$m_{jk} = \frac{1}{u_{kk}} \left(a_{jk} - \sum_{s=1}^{k-1} m_{js} u_{sk} \right) \qquad j = k+1,\dots\dots,n;\ k \geq 2$$

LU Factorization Example

Following a Gauss Elimination type strategy

(from Prof. Amrik Sen's notes)

1	0	0	0		1	1	0	3
0	1	0	0		2	1	-1	1
0	0	1	0		3	-1	-1	2
0	0	0	1		-1	2	3	-1

I ***A***

1	0	0	0		1	1	0	3
2	1	0	0		0	-1	-1	-5
3	0	1	0		0	-4	-1	-7
-1	0	0	1		0	3	3	2

$E_2 - 2E_1$ $m_{21}=2$

$E_3 - 3E_1$ $m_{31}=3$

$E_4 - (-1)E_1$ $m_{41}=-1$

1	0	0	0		1	1	0	3
2	1	0	0		0	-1	-1	-5
3	4	1	0		0	0	3	13
-1	-3	0	1		0	0	0	-13

L ***U***

$E_3 - 4E_2$ $m_{32}=4$

$E_4 - (-3)E_2$ $m_{42}=-3$ $m_{43}=0$

LU Factorization Example

Following a Gauss Elimination type strategy

$$\begin{pmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 6 & 2 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & & 3 & 5 & 2 \\ 0 & 1 & 0 & & 0 & 8 & 2 \\ 0 & 0 & 1 & & 6 & 2 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & & 3 & 5 & 2 \\ 0 & 1 & 0 & & 0 & 8 & 2 \\ 0 & 0 & 1 & & 6 & 2 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & & 3 & 5 & 2 \\ 0 & 1 & 0 & & 0 & 8 & 2 \\ 2 & 0 & 1 & & 0 & -8 & 4 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 & & 3 & 5 & 2 \\ 0 & 1 & 0 & & 0 & 8 & 2 \\ 2 & -1 & 1 & & 0 & 0 & 6 \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{pmatrix}}_{U}$$

Solving a set of Linear Equations using *LU* Factorization

$$Ax = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}}_{L}\underbrace{\begin{pmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{pmatrix}}_{U} x = b = \begin{pmatrix} 8 \\ -7 \\ 26 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 1 \end{pmatrix}}_{L} y = b = \begin{pmatrix} 8 \\ -7 \\ 26 \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}}_{y} = \begin{pmatrix} 8 \\ -7 \\ 3 \end{pmatrix}$$

$$Ux = y \quad \Rightarrow \begin{pmatrix} 3 & 5 & 2 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{pmatrix} x = \begin{pmatrix} 8 \\ -7 \\ 3 \end{pmatrix} \quad \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 1/2 \end{pmatrix}$$

Solving a set of Linear Equations using *LU* Factorization *(Prof. Amrik Sen's example)*

$$Ax = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix}}_{U} x = b = \begin{pmatrix} 4 \\ 1 \\ -3 \\ 4 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{pmatrix}}_{L} y = b = \begin{pmatrix} 4 \\ 1 \\ -3 \\ 4 \end{pmatrix} \Rightarrow \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}}_{y} = \begin{pmatrix} 4 \\ -7 \\ 13 \\ -13 \end{pmatrix}$$

$$Ux = y \quad \Rightarrow \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix} x = \begin{pmatrix} 4 \\ -7 \\ 13 \\ -13 \end{pmatrix} \qquad \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

Orthogonal Basis and Gram-Schmidt Orthogonalization

Two vectors $\vec{u}_1$ and $\vec{u}_2$ are ***orthogonal*** if and only if $< \vec{u}_1, \vec{u}_2 >= 0$

The vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_m \in R^n$ are ***orthonormal*** if and only if $< \vec{u}_i, \vec{u}_j >= \delta_{ij} = 1 \ \ i = j$

$= 0 \ \ i \neq j$

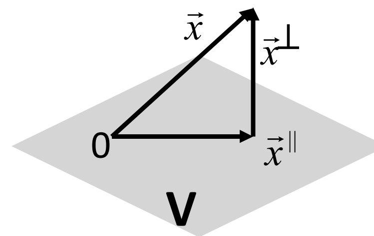
Example: $\vec{e}_1, \vec{e}_2, ..., \vec{e}_n \in R^n$ are orthonormal

Properties of orthonormal vectors

1. Orthonormal vectors are (automatically) linearly independent
2. Orthonormal vectors $\vec{u}_1, \vec{u}_2, ..., \vec{u}_n \in R^n$ form a basis in R^n

Orthogonal projection and orthogonal complement:

Let $\overrightarrow{x} \in \mathbb{R}^n$ and a subspace V of $\mathbb{R}^n$. Then we can wrtie $\overrightarrow{x} = \overrightarrow{x}^{||} + \overrightarrow{x}^{\perp}$, where $\overrightarrow{x}^{||} \in V$ and $\overrightarrow{x}^{\perp} \in V^{\perp}$. The above representation is **unique**.



Here $V^{\perp} = \{\overrightarrow{x} \in \mathbb{R}^n : \langle \overrightarrow{v}, \overrightarrow{x} \rangle = 0, \ \ \forall \overrightarrow{v} \in V\}$. The transformation $T(\overrightarrow{x}) = \text{proj}_V \overrightarrow{x} = \overrightarrow{x}^{||}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is linear. $V^{\perp} = \text{Ker}(T)$.

How do we compute $\overrightarrow{x}^{||}$?

Consider an orthonormal basis of V: $\overrightarrow{u}_1, \overrightarrow{u}_2, \cdots, \overrightarrow{u}_m \in V$ which is a subspace of $\mathbb{R}^n$. Then

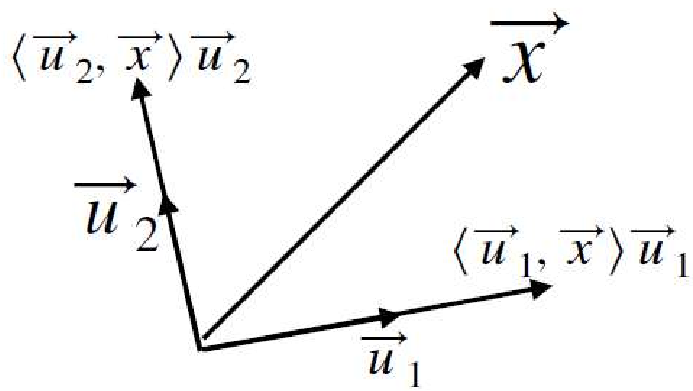
$$\overrightarrow{x}^{||} = \langle \overrightarrow{u}_1, \overrightarrow{x} \rangle \overrightarrow{u}_1 + \cdots + \langle \overrightarrow{u}_m, \overrightarrow{x} \rangle \overrightarrow{u}_m; \quad \forall \overrightarrow{x} \in \mathbb{R}^n.$$

Consequently, consider an orthonormal basis of $\mathbb{R}^n$: $\overrightarrow{u}_1, \overrightarrow{u}_2, \cdots, \overrightarrow{u}_n$. Then any $\overrightarrow{x} \in \mathbb{R}^n$,

$$\overrightarrow{x} = \langle \overrightarrow{u}_1, \overrightarrow{x} \rangle \overrightarrow{u}_1 + \cdots + \langle \overrightarrow{u}_n, \overrightarrow{x} \rangle \overrightarrow{u}_n.$$

$\vec{x}^{\perp}$ Component of ***x*** perpendicular to ***V***

$\vec{x}^{\parallel}$ Projection of ***x*** on ***V***



Properties of $V^{\perp}$, the Orthogonal Complement of V :

Consider a subspace $V \in \mathbb{R}^n$. Then, we have the following important results about $V^{\perp}$, the orthogonal complement of V

1. $V^{\perp}$ is a subspace of $\mathbb{R}^n$.
2. $V \cap V^{\perp} = \{0\}$.
3. $\dim(V) + \dim(V^{\perp}) = n$.
4. $(V^{\perp})^{\perp} = V$.

Example:

Consider the subspace *V*=*Im*(*A*) of $\mathbb{R}^4$ where *A*= $\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$. Find $\vec{x}^{\parallel}$ for $\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 7 \end{pmatrix}$

Solution: Recall that the column space of A is $Im(A)$. It can be easily checked that the column vectors of A are orthogonal by taking their scalar product. Thus we can construct an orthonormal basis of $Im(A)$. The basis vectors $\vec{u}_1, \vec{u}_2$ are -

$$\vec{u}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \ \vec{u}_2 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix} \text{ and } \vec{x}^{\parallel} = \langle \vec{u}_1, \vec{x} \rangle \vec{u}_1 + \langle \vec{u}_2, \vec{x} \rangle \vec{u}_2 = 6\vec{u}_1 + 2\vec{u}_2 = \begin{pmatrix} 4 \\ 2 \\ 2 \\ 4 \end{pmatrix}$$

What would be $x^{\perp}$?

Verify that $\vec{x} - \vec{x}^{\parallel} \perp \vec{u}_1, \vec{u}_2$ perpendicular to both $\vec{u}_1$ and $\vec{u}_2$

Why are orthonormal basis vectors useful?

1. We know that if we have some basis $\vec{v}_1, \vec{v}_2, \ldots\ldots, \vec{v}_n$ of an n-dimensional vector space W, then any vector $\vec{x} \in W$ can be written as $\vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \ldots\ldots + \alpha_n \vec{v}_n$ (i.e. as a linear combination of the basis vectors). However, there is no first-principles or convenient way of finding the unique coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ except by explicit guesswork calculations. On the contrary, if we do have an orthonormal basis set $\vec{u}_1, \vec{u}_2, \ldots\ldots\ldots \vec{u}_n$ then any vector can be written as a linear combination of this orthonormal basis set as

$$\vec{x} = \beta_1 \vec{u}_1 + \beta_2 \vec{u}_2 + \ldots\ldots + \beta_n \vec{u}_n$$

where the coefficients $\beta_i = \langle \vec{u}_i, \vec{x} \rangle, \forall i = 1, 2, \ldots\ldots, n$ are **uniquely determined**.

2. Orthogonality guarantees linear independence.

Orthogonal Transformation: A linear transformation ***T*** : ***V*** → ***V*** in $\mathbb{R}^n$ that preserves the inner product. For each pair ***u***, **v** of elements of ***V***, we will then have -

$$\langle u, v \rangle = \langle Tu, Tv \rangle$$

Why are orthogonal transformations useful?

1. Orthogonal transformations are metric preserving transformations, i.e. if $\boldsymbol{T}: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal, then $\left|| \boldsymbol{T}(\vec{x}) |\right| = \left|| \vec{x} |\right|, \forall \vec{x} \in \mathbb{R}^n$.

   (*length preserving transformation*)

   (Note that if $T(\vec{x}) = A\vec{x}$ is an orthogonal transformation, then we say that A is an orthogonal matrix. For such an *A,* we have that $A^TA=I$ or $A^{-1}=A^T$)

2. Orthogonal transformations are angle preserving transformations for orthogonal vectors.

   For example, if $\vec{u} \perp \vec{w}$, then $T(\vec{u}) \perp T(\vec{w})$