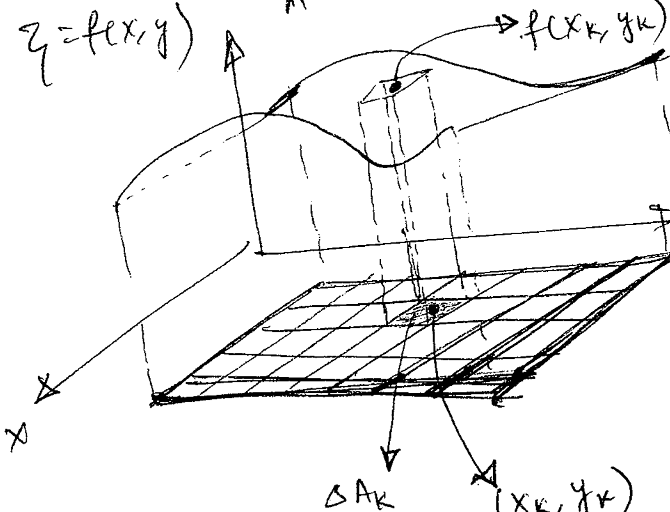
$(13.1) Double Integrals

Amerik's Copy!

(I) Over Rectangles

Let $f(x,y)$ be defined ~~by~~ on a rectangular region, R

$R: a \le x \le b, \quad c \le y \le d$



Construct, $S_n = \sum_{K=1}^{n} f(x_K, y_K)\,\Delta A_K \equiv \sum_{j=1}^{n}\sum_{i=1}^{n} f(x_i, y_j)\,\Delta A_{ij}$

What happens if (1) f is continuous over R?

& (1) $\Delta A_K \to 0$ (why?)

$$\iint_R f(x,y)\,dA \equiv \iint_R f(x,y)\,dx\,dy$$

$$= \lim_{\Delta A_K \to 0} \sum_{K=1}^{n} f(x_K, y_K)\,\Delta A_K$$

* Continuity of f over R is a Sufficient condition for the existence of the double integrals, but not a necessary one.

The limit may ~~may~~ exist for many discont. f^n also.

(II) Properties of Double Integrals.

(1) $\iint_R K f(x,y)\,dA = K \iint_R f(x,y)\,dA$; K const.

(2) $\iint_R (f \pm g)\,dA = \iint_R f\,dA \pm \iint_R g\,dA$.

(3) $\iint_R f\,dA \geq 0 \Leftarrow f \geq 0$ on R

(4) $\iint_R f\,dA \geq \iint_R g\,dA \Leftarrow f \geq g$ on R

(5) $\iint f\,dA = \iint_{R_1} f\,dA + \iint_{R_2} f\,dA$.

$R = R_1 \cup R_2$
where R_1 & R_2 are disjoint.

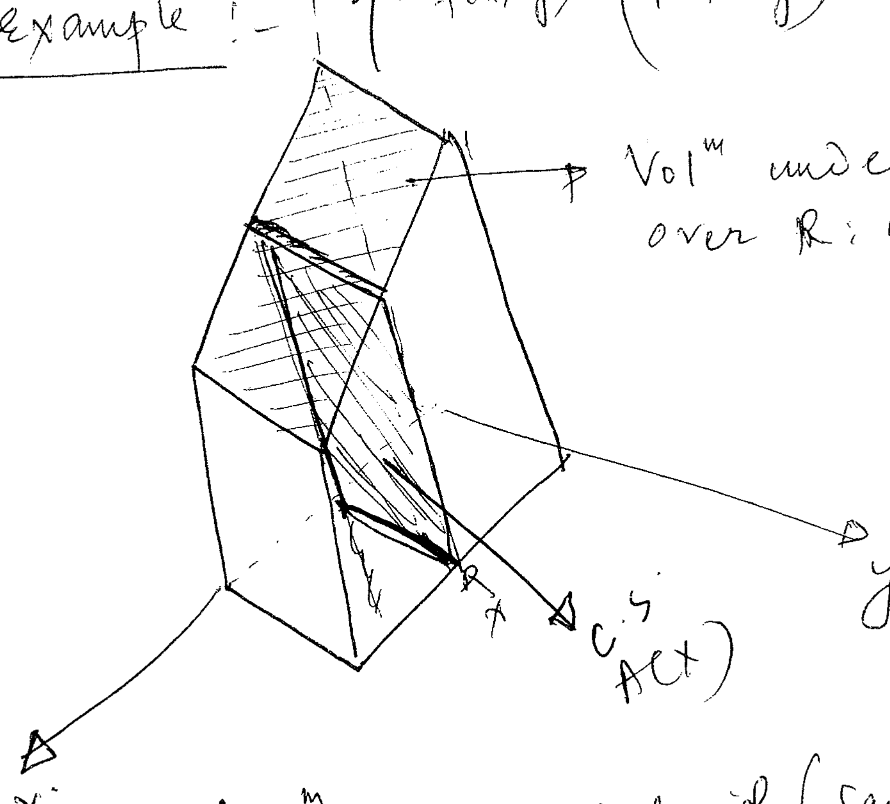
(III) Double Integrals as Volumes

* When $f(x,y) > 0$; we may interpret the double integral of f over a rectangular region R as the volume of the solid prism bounded below by R and above by the surface $z = f(x,y)$.

* Each term $(f(x_K, y_K)\,\Delta A_K)$ in the sum $S_n = \sum_{K=1}^{n} f(x_K, y_K)\Delta A_K$ is the vol^m of a vertical rectangular prism that approximates the vol^m of the portion of the solid that stands directly above the base ΔA_K.

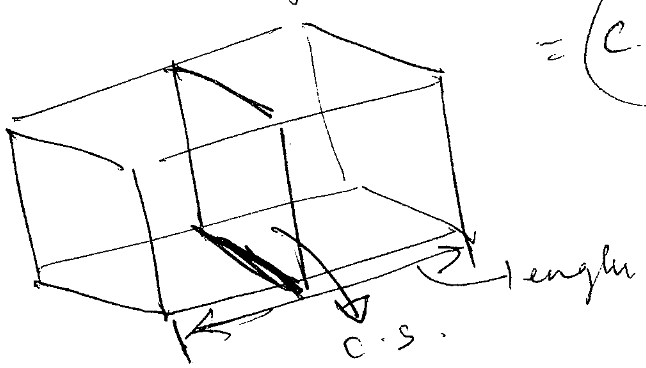
i.e. $Vol^m = \iint_R f(x,y)\,dA$.

Example :- $z = f(x,y) = (4-x-y)$



$\rightarrow$ Vol^m under this plane over $R: 0 \le x \le 2,\ 0 \le y \le 1$?

Vol^m of a cuboid (say)
$= (\text{c.s. area}) \times$ length b/c c.s. is of const. area/ht



In the same spirit;

Area of the c.s. @ x is $A(x) = \int_{y=0}^{1} (4-x-y)\,dy$

$= \left.(4-x)y - \frac{y^2}{2}\right|_0^1$

$= (4-x) - \frac{1}{2}$

$= \frac{8-1}{2} - x$

$= \left(\frac{7}{2} - x\right)$

Now;

$$\text{Vol}^m \approx \sum_{\substack{\forall i \text{ s.t.} \\ 0 \le x_i \le 2}} (A(x_i)) \times \underbrace{(x_{final} - x_{init})}_{\Delta x_i}$$

$$\approx \int_{x=0}^{2} A(x)\,dx = \int_0^2 \left(\frac{7}{2} - x\right) dx = 5$$

Similarly, we could have taken c.s. $//^{lel}$ to x-axis and arrived at the same result.

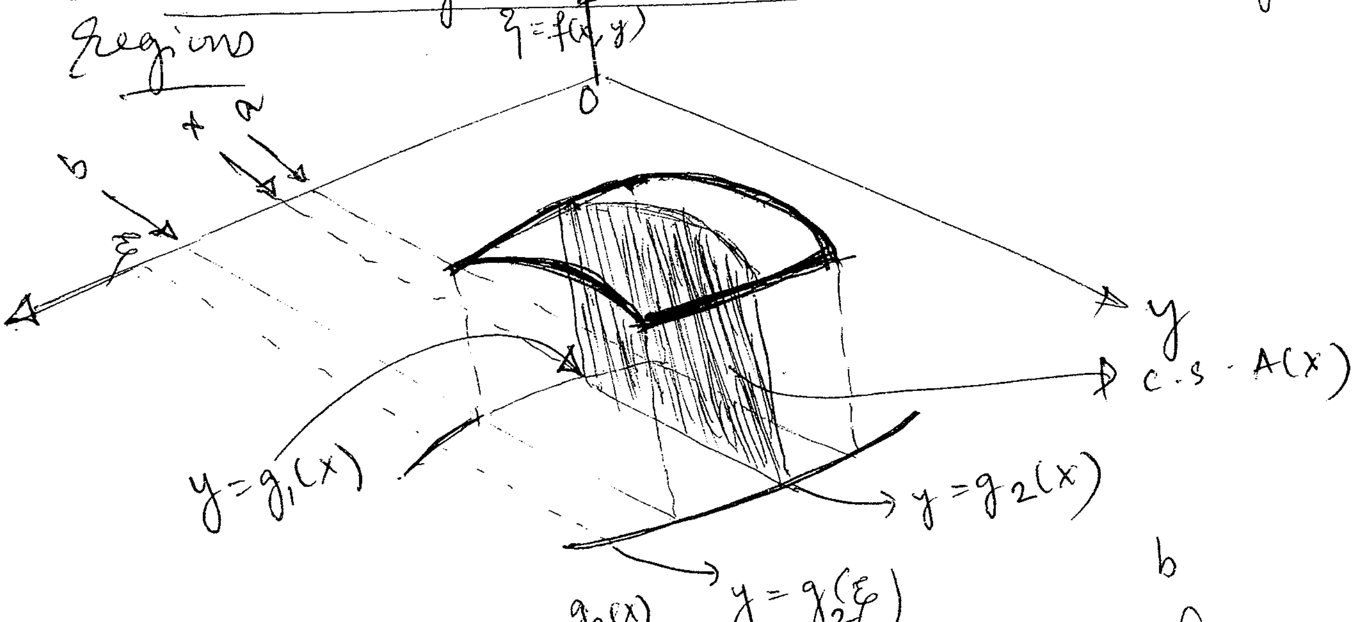
$\Rightarrow$ leads us to <u>Fubini's Th^m (I)</u>

If $f(x,y)$ is continuous on the <u>rectangular region</u> $R: a \le x \le b,\ c \le y \le d$; then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

<u>Note</u>:- The Double integral is consistent when we swap the order of integration.

(IV) <u>Double Integral over Bdd non-rectangular regions</u>



c.s.; $A(x) = \int_{y=g_1(x)}^{g_2(x)} f(x,y)\,dy$ and $V = \int_a^b A(x)\,dx.$
@ x

$$\therefore V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx .$$

Similarly;

$$V = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

Fubini's Th^m (II)

Let $f(x,y)$ be continuous on R.

(i) If R: $a \le x \le b$, $g_1(x) \le y \le g_2(x)$ with g_1 and g_2 continuous on $[a,b]$; then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx .$$

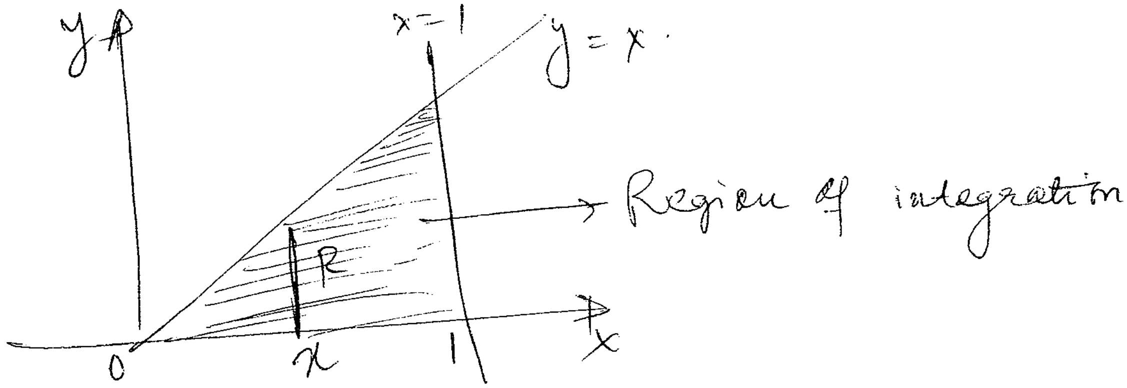
(ii) If R: $c \le y \le d$, $h_1(y) \le x \le h_2(y)$; with h_1 & h_2 cont. on $[c,d]$; then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy .$$

eg Calculate

$$\iint \frac{\sin x}{x}\, dA$$

R; $\triangle$ in the xy-plane bdd by x-axis, $y=x$ & $x=1$



Step (1) :- Draw the region

Step (2) :- Choose your order of integration.

(a) Let's integrate along the y - dir^n first

$$A(x) = \int_{y=0}^{y=x} \frac{\sin x}{x} \, dy$$

$$= \frac{\sin x}{x} \, y \Big|_0^x$$

$$= \frac{x \sin x}{x} - 0$$

$$= \sin x$$

(b) Now integrate along the x- dir^n

$$\iint_R \frac{\sin x}{x} \, dA = \int_{x=0}^{1} \sin x \, dx = -\cos x \Big|_0^1$$

$$= -\cos 1 - (-1)$$

$$= 1 - \cos 1$$

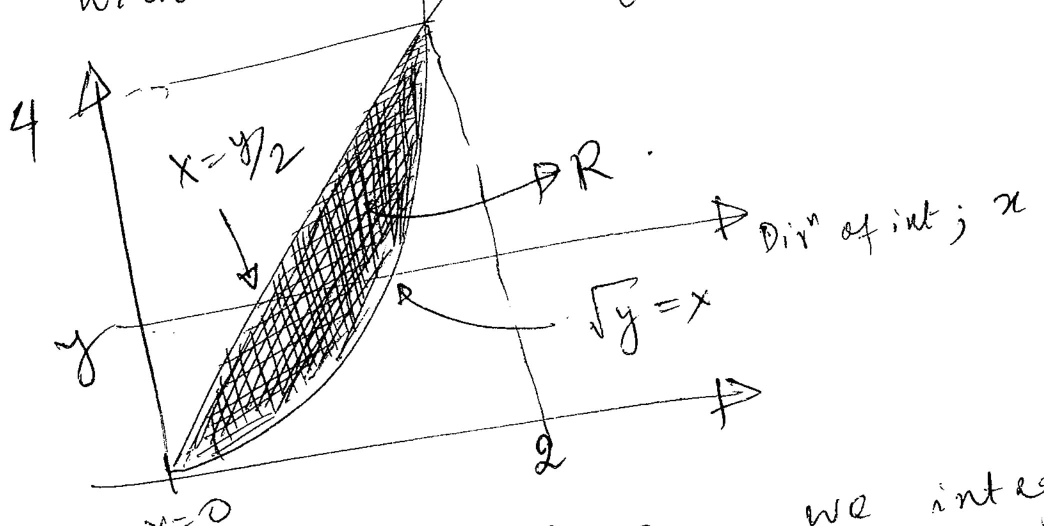
* Would it have been convenient, had we chosen the other order of integration?

So; choosing the order of integr^n is a matter of convenience.

eg Sketch the region of integration for the integral

$$\int_0^2 \int_{x^2}^{2x} (4x+2)\,dy\,dx \quad — ①$$

And write an equivalent integral with the order of integration reversed.



Step (1) :- Which dir^n are we integrating 1st in (1).

Ans :- y!

Step (2) :- Sketch $y = x^2$ and $y = 2x$.

Step (3) :- What are the limits of integration in the other dir^n.

Ans :- $x = 0$ & $x = 2$

$y = 2(2) = 4$

$y = x^2 = 4$

Now; ① let's integrate along x - first → ∴ Draw the dir^n $\parallel^{lel}$ to x - axis.

② Spot the bdy of integ^n for some y;

(3) so $A(y) = \int_{x=\frac{y}{2}}^{x=\sqrt{y}} (4x+2)\,dx$

(4) Compute the limits for y

$$V = \int_{y=0}^{4} \left(\int_{x=y/2}^{x=\sqrt{y}} (4x+2)\,dx \right) dy$$

§ (13.2) Areas, Moments and Centers of Mass.

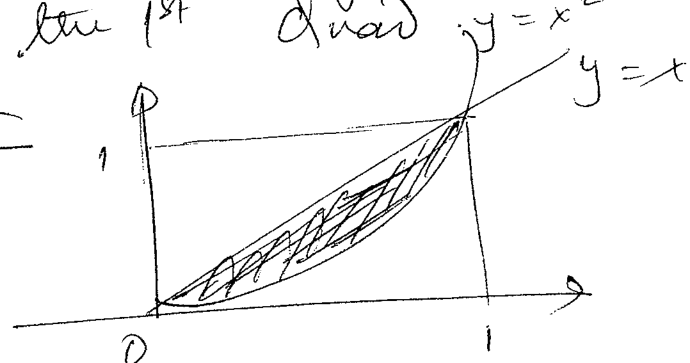
Def^n :- Area of a closed, bdd planar region R is

(I) $A = \iint_R dA \approx \sum_{k=1}^{n} \Delta A_k$.

(II) Avg. value of f over R $= \frac{1}{\text{area of } R} \iint_R f\,dA$

eg: Find the area of the region R bounded by $y = x$ and $y = x^2$ in the 1st quad.

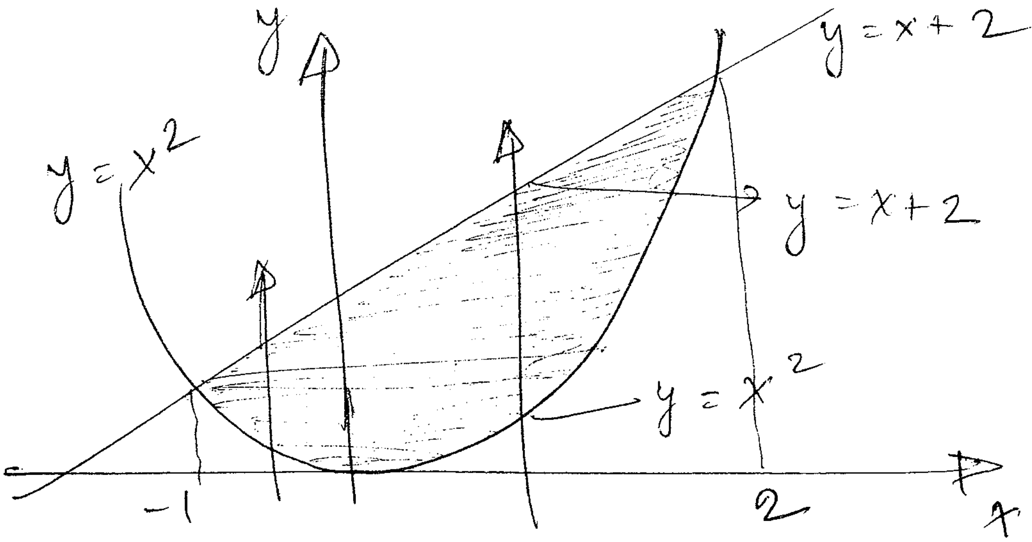
Ans :-



$$A = \int_0^1 \int_{x^2}^{x} dy\,dx = \frac{1}{6}.$$

eg Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$?

(Hint :- Use property (5) from § (13.1) (II)).



Method (1) :-

(1) Choose the initial dir^n of integ^n $\parallel^{lel}$ to y

$$A(x) = \int_{y=x^2}^{y=x+2} dy$$

$$= \left(y \right)_{x^2}^{x+2}$$

$$= x+2-x^2$$

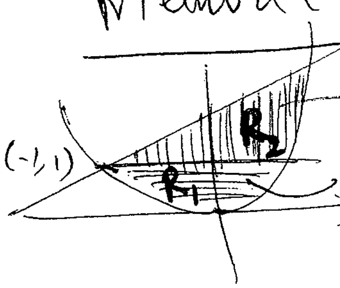
Solve for y & x from

(2) $y = x^2$
$y = x+2$
$\Rightarrow x^2 - x - 2 = 0.$
$\Rightarrow x^2 - 2x + x - 2 = 0$
$\Rightarrow x(x-2) + 1(x-2) = 0$
$\Rightarrow (x+1)(x-2) = 0$
$x = -1, 2$

$$A = \int_{-1}^{2} (\cancel{A}(x))\, dx$$

$$= \int_{-1}^{2} (x+2-x^2)\, dx = \frac{x^2}{2} + 2x - \frac{x^3}{3} \Big|_{-1}^{2} = \frac{9}{2}$$

Method (2) :- Now compute the same integral by reversing the order of integration.



$$\int_1^4 \int_{y-2}^{\sqrt{y}} dx\,dy$$

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy$$

$$A = \iint_{R_1} dA + \iint_{R_2} dA \;\; \frac{\text{Should}}{\text{get}} \;\; 9/2$$

6/29/10

Density :- $\delta(x,y)$

Mass : $M = \iint \delta(x,y)\, dA$

1^{st} moments :- $M_x = \iint y\,\delta(x,y)\, dA$ } Balancing moments

$M_y = \iint x\, \delta(x,y)\, dA$

Center of mass :- $\bar{x} = \frac{M_y}{M}$

$\bar{y} = \frac{M_x}{M}$

Moments of Inertia (2^{nd} moments) } Turning moments

About the X-axis :- $I_x = \iint y^2 \delta(x,y)\, dA$

About the Y-axis :- $I_y = \iint x^2 \delta(x,y)\, dA$

polar moment

About a line L :- $I_L = \iint r^2(x,y)\, \delta(x,y)\, dA$

About origin :- $I_0 = \iint (x^2+y^2)\,\delta(x,y)\, dA = (I_x + I_y)$ where $r(x,y)$ = distance from (x,y) to L

Radii of gyration :-

About the X-axis : $R_x = \sqrt{\frac{I_x}{M}}$

" Y-axis :- $R_y = \sqrt{\frac{I_y}{M}}$

" origin :- $R_0 = \sqrt{\frac{I_0}{M}}$

Application :-

K.E. mass $= \frac{1}{2} m v^2$ (1)

K.E. shaft $= \frac{1}{2} I \omega^2$

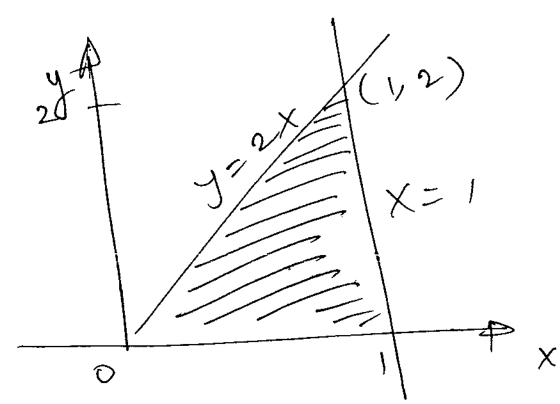
A shaft's moment of inertia is analogous to the locomotive's mass. What makes the locomotive hard to start or stop is its mass; what makes the shaft hard to start or stop (rotating) is its M.O.I. . The M.O.I. takes into account not only the mass but also its distribution.

(2) The M.O.I. also plays a role in determining how much a horizontal beam will bend under load.

The greater the value of I; the stiffer the beam and the less it will bend under a given load.

e.g. A thin plate covers the triangular region bounded by the x-axis and the lines $x = 1$ and $y = 2x$ in the 1st quadrant. The plate's density at the point (x, y) is $\delta(x,y) = 6x + 6y + 6$. Find the plate's mass, 1st moments, center of mass, M.O.I. and radii of gyration about the co-ordinate axes?

Soln :-



plate's mass,

$$M = \int_0^1 \int_0^{2x} \delta(x,y)\,dy\,dx = \int_0^1 \int_0^{2x} (6x + 6y + 6)\,dy\,dx = \cdots = 14$$

$$M_x = \int_0^1 \int_0^{2x} y\,\delta(x,y)\,dy\,dx = \cdots = 11$$

$$M_y = \int_0^1 \int_0^{2x} x\,\delta(x,y)\,dy\,dx = 10$$

C.O.M. :- $\bar{x} = \frac{M_y}{M} = \frac{10}{14} = \frac{5}{7}$; $\bar{y} = \frac{M_x}{M} = \frac{11}{14}$.

M.O.I.

$$I_x = \int_0^1 \int_0^{2x} y^2 \delta(x,y)\, dy\, dx$$

$$= \ldots = 12$$

$$I_y = \int_0^1 \int_0^{2x} x^2 \delta(x,y)\, dy\, dx = \frac{39}{5}$$

$$I_0 = I_x + I_y = \frac{99}{5}.$$

Radii of gyration :-

$$R_x = \sqrt{\frac{I_x}{M}} = \sqrt{6/7}$$

$$R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{39}{70}}$$

$$R_0 = \sqrt{\frac{I_0}{M}} = \sqrt{\frac{99}{70}}$$

Centroids of Geometric Figures

* When the density of an object is constant, it cancels out of the numerator & denominator of the formulae $\bar{x} = \frac{M_y}{M}$ and $\bar{y} = \frac{M_x}{M}$.

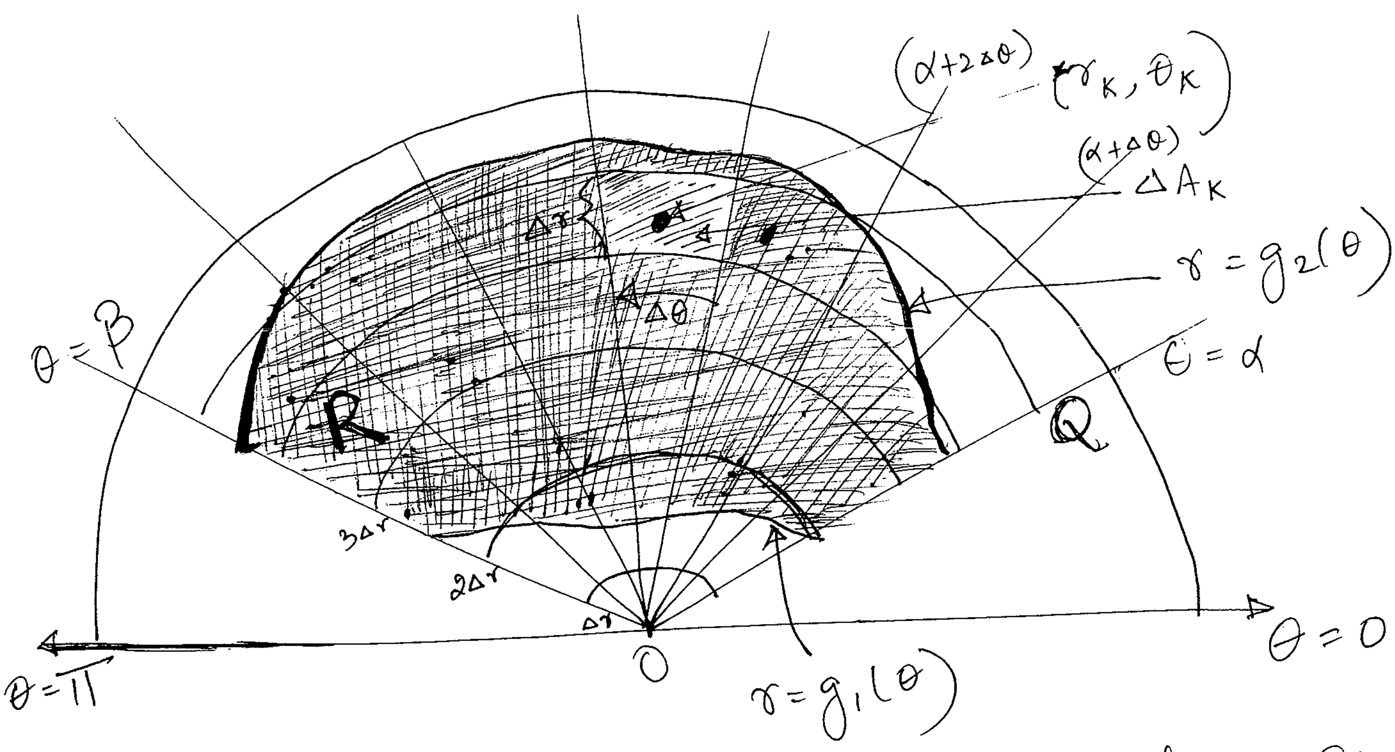
∴ When δ is constant, the location of the C.O.M. becomes a feature of the object's shape and not of the material of which it is made. In such cases, engineers may call the C.O.M., the $\boxed{\text{Centroid}}$ of the shape.

(13.3) Double integrals in Polar Form

(a matter of convenience).

In polar co-ordinates, the natural shape is a "polar rectangle" whose sides have constant r- and θ- values.

Suppose that a function $f(r,\theta)$ is defined over a region R that is bounded by the rays $\theta=\alpha$ and $\theta=\beta$ and by the continuous curves $r=g_1(\theta)$ and $r=g_2(\theta)$. Suppose also that $0 \le g_1(\theta) \le g_2(\theta) \le a$ for every value of θ between α and β. Then R lies in a fan-shaped region Q defined by the inequalities $0 \le r \le a$ and $\alpha \le \theta \le \beta$



We cover Q by a grid of circular arcs and rays. These arcs and rays partition Q into small patches called "polar rectangles"

We number the polar rectangles that lie inside R as $\Delta A_1, \Delta A_2, \ldots, \Delta A_n$.

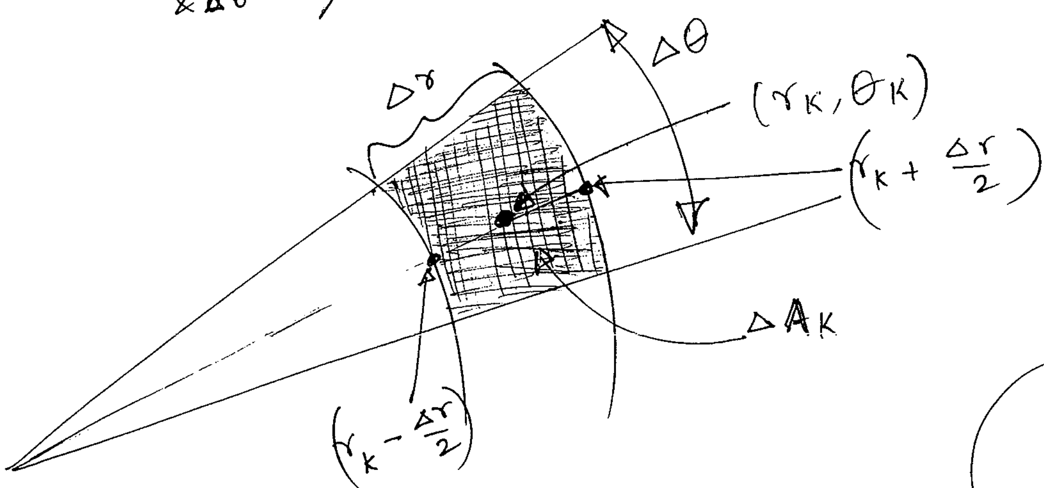
Let (r_K, θ_K) be the center of the polar rectangle whose area is ΔA_K.

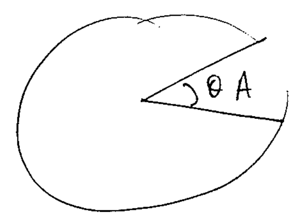
Construct,

$$S_n = \sum_{K=1}^{n} f(r_K, \theta_K)\, \Delta A_K \quad \text{——} (1)$$

If f is continuous over R then

$$\lim_{\substack{n \to \infty \\ (\text{i.e. } \Delta r \to 0 \\ \& \Delta\theta \to 0)}} S_n = \iint_R f(r,\theta)\, dA$$





$$\Delta A_K = \frac{1}{2}\Delta\theta\left(r_K + \frac{\Delta r}{2}\right)^2 - \frac{1}{2}\Delta\theta\left(r_K - \frac{\Delta r}{2}\right)^2$$

$$= r_K\, \Delta r\, \Delta\theta$$

$$\frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$

$$A = \frac{1}{2}\theta r^2$$

$\therefore$ from (1) we get

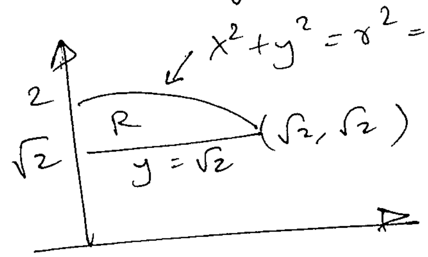
$$S_n = \sum_{K=1}^{n} f(r_K, \theta_K)(r_K\, \Delta r\, \Delta\theta) \xrightarrow{n \to \infty} \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

i.e. $\displaystyle\iint_R f(r,\theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$

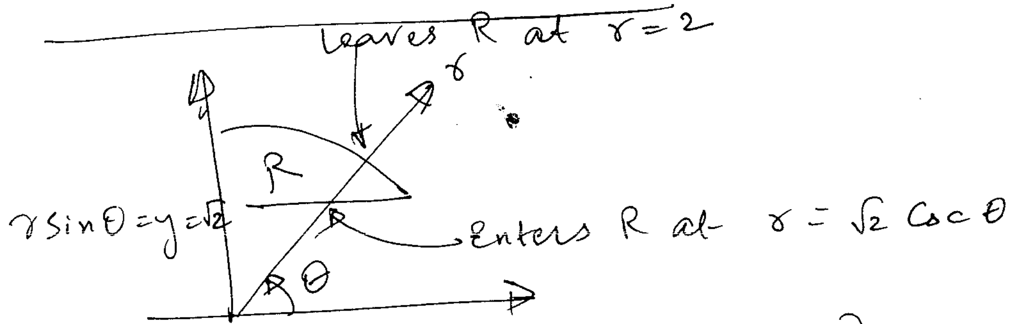
How to integrate in polar Co-ordinates:

$$\iint_R f(r,\theta)\, dA = \iint_R f(r,\theta)\, r\, dr\, d\theta$$

(1) Sketch region & label bounding curves.

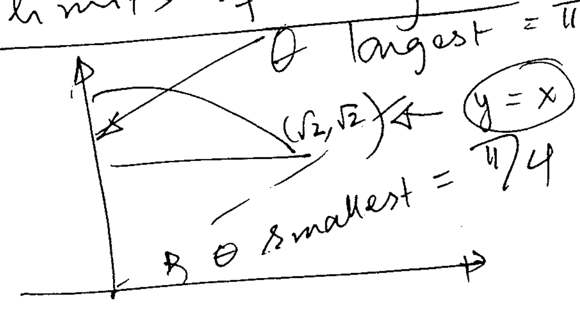


(2) r-limits of integ^n



* r-limits will usually depend on θ. th al the $\nearrow r$ makes with +ve x-axis.

(3) θ-limits of integ^n



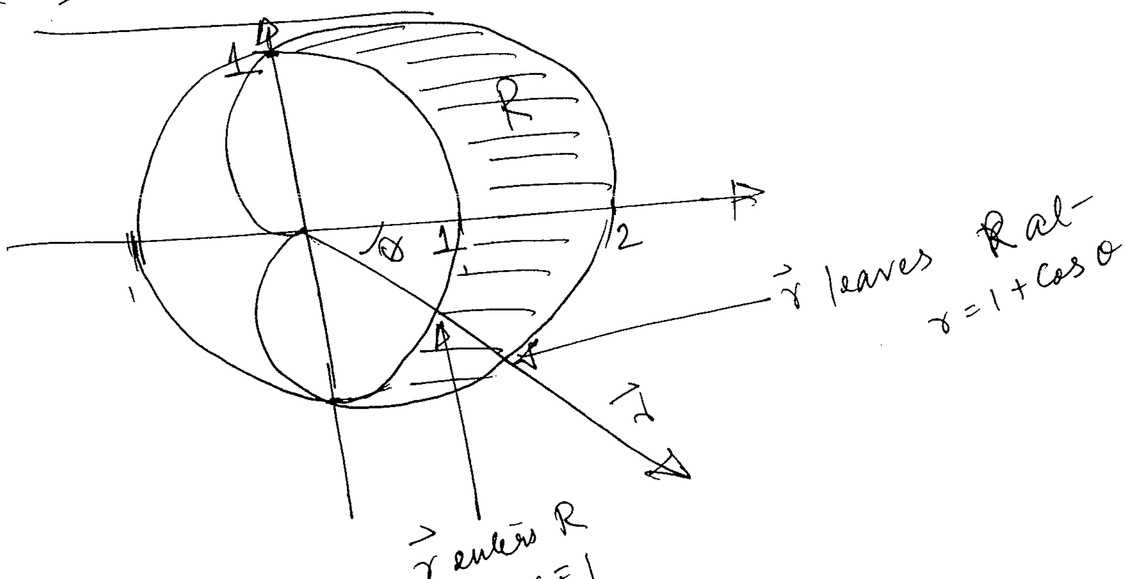
$$\therefore \iint_R f(r,\theta)\, dA = \int_{\pi/4}^{\pi/2}\int_{\sqrt{2}\csc\theta}^{r=2} f(r,\theta)\, r\, dr\, d\theta$$

Area of a closed & bdd region R in polar - co-ordinate plane is

$$A = \iint_R r\,dr\,d\theta = \iint_R dA.$$

eg Find the limits of integration for $\iint f(r,\theta)\,r\,dr\,d\theta$ over R that lies inside the cardoid $r = 1+\cos\theta$ & outside the circle $r=1$

Soln :- (I) Sketch R.



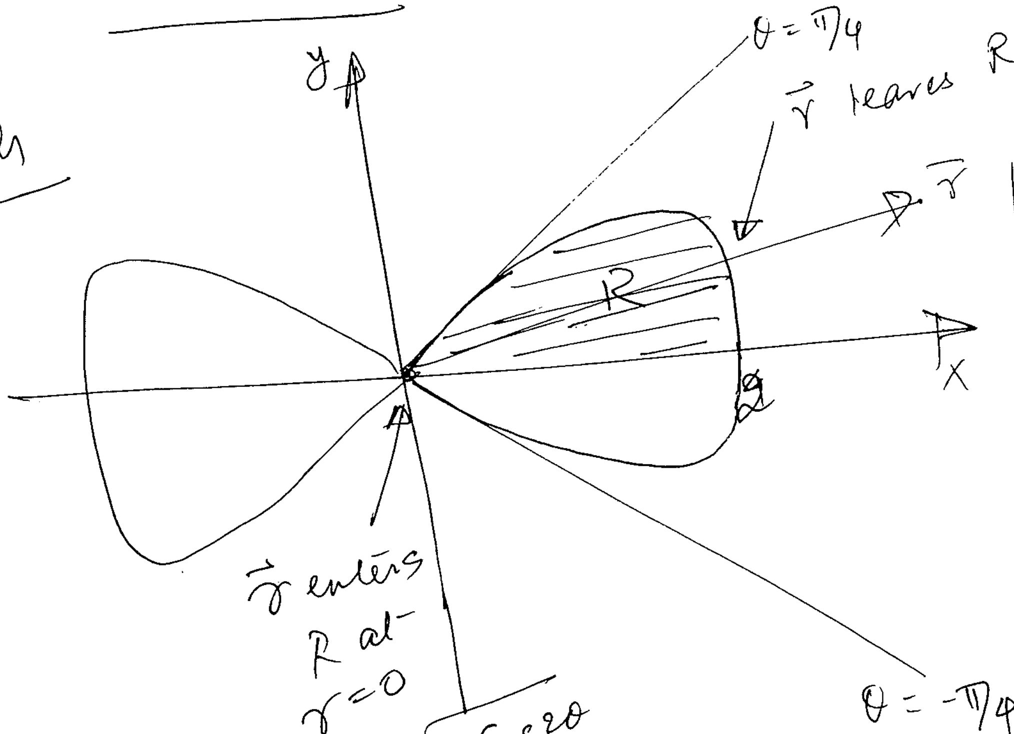
(II) r- limits : $r=1$ to $r = 1+\cos\theta$

(III) θ- limits :- $\theta = -\pi/2$ to $\pi/2$

$$\therefore \int_{-\pi/2}^{\pi/2}\int_{1}^{(1+\cos\theta)} f(r,\theta)\,r\,dr\,d\theta.$$

eg. Find the area enclosed by the lemniscate $r^2 = 4\cos 2\theta$.

Ans ① Sketch



$\theta = 0 \quad r^2 = 4$
$r = 2$.

$\theta = \pi/2 \quad \cos\pi = -1$
$r^2 = -4$
DNE.

$\theta = \pi/4, -\pi/4$
$\cos 2\theta = 0$.
$r = 0$

$\theta = \frac{\pi}{8}$.
$\cos 2\theta = \cos \pi/4 = \frac{1}{\sqrt{2}}$
$r^2 = \frac{2 \times 2}{\sqrt{2}}$
$r^2 = 2\sqrt{2}$
$r = (2 \cdot 2^{1/2})^{1/2}$
$= 2^{\frac{3}{2}\cdot\frac{1}{2}}$
$= 2^{3/4}$
$= 2$

$$A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4}\left(\frac{r^2}{2}\right)\Big|_0^{\sqrt{4\cos 2\theta}}$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta$$

$$= 4\sin 2\theta\Big|_0^{\pi/4} = 4.$$

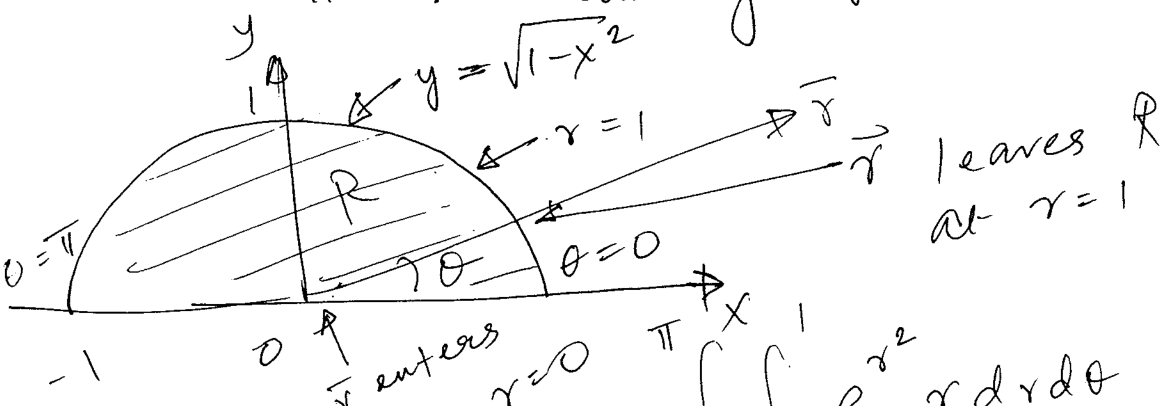
Changing Cartesian Integral to Polar Integral

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

G — region of integ^n in polar co-ordinates

e.g. Evaluate :- $\displaystyle\iint_R e^{x^2+y^2}\,dy\,dx$

R :- Semi circular reg^n bdd by x-axis and $y = \sqrt{1-x^2}$



Let $r^2 = u$
$2r\,dr = du$

$$\int_{\theta=0}^{\pi}\int_{r=0}^{1} e^{r^2}\, r\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{u=0}^{1} e^{u}\,\cancel{r}\,\frac{du}{2\cancel{r}}\,d\theta$$

$$= \frac{1}{2}\int^{\pi} e^{r^2}\,d\theta = \frac{\pi}{2}(e-1)$$

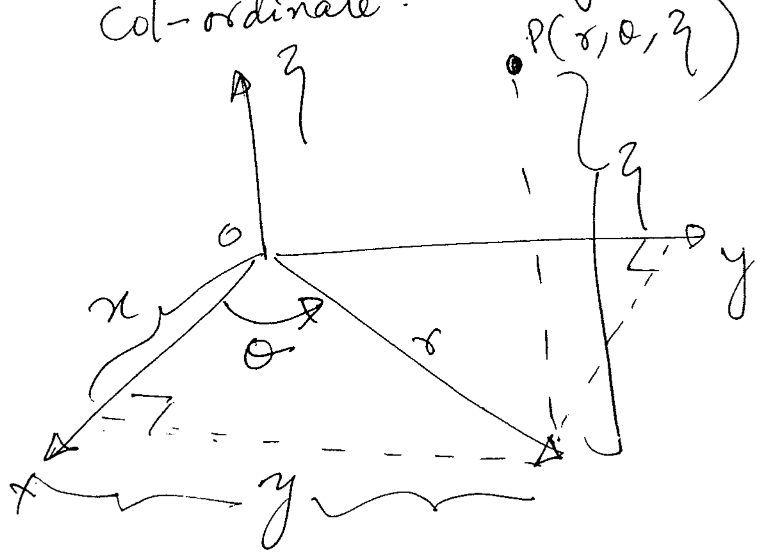
6/30/2010

§(10.7) Cylindrical & Spherical Co-ordinates

Cylindrical Co-ordinates (r, θ, z) represent a pt P in space by ordered triples (r, θ, z) where

(i) r and θ are polar Co-ordinates for the vertical projection of P on the xy-plane

(ii) z is the Rectangular Vertical Co-ordinate.



$$\therefore \boxed{\begin{gathered} x = r\cos\theta;\ y = r\sin\theta;\ z = z \\ r^2 = x^2 + y^2;\ \tan\theta = \left(\frac{y}{x}\right) \end{gathered}}$$

(a) Here $r = a \Rightarrow$ a cylinder about the z-axis (not just a circle)

(b) $\theta = \theta_0$ describes the plane that contains the z-axis & makes an $\angle^{le}$ θ_0 w/ +ve x-axis.

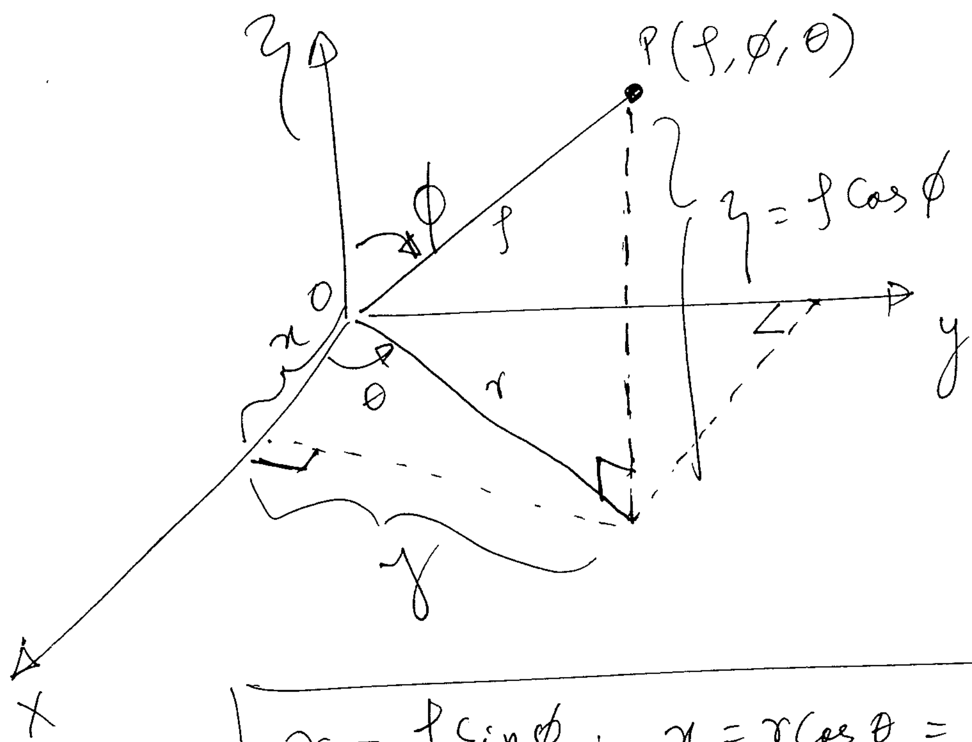
(c) $z = z_0$ describes a plane $\perp$ to the z-axis.

Spherical Co-ordinates (ρ, ϕ, θ) represent a point P in space by ordered triples (ρ, ϕ, θ) where

(i) ρ is the distance from P to origin

(ii) ϕ is the $\angle^{le}$ $\overline{OP}$ makes with $+z$ axis $(0 \le \phi \le \pi)$

(iii) θ is the same $\angle^{le}$ as in Cylindrical Co-ordinates



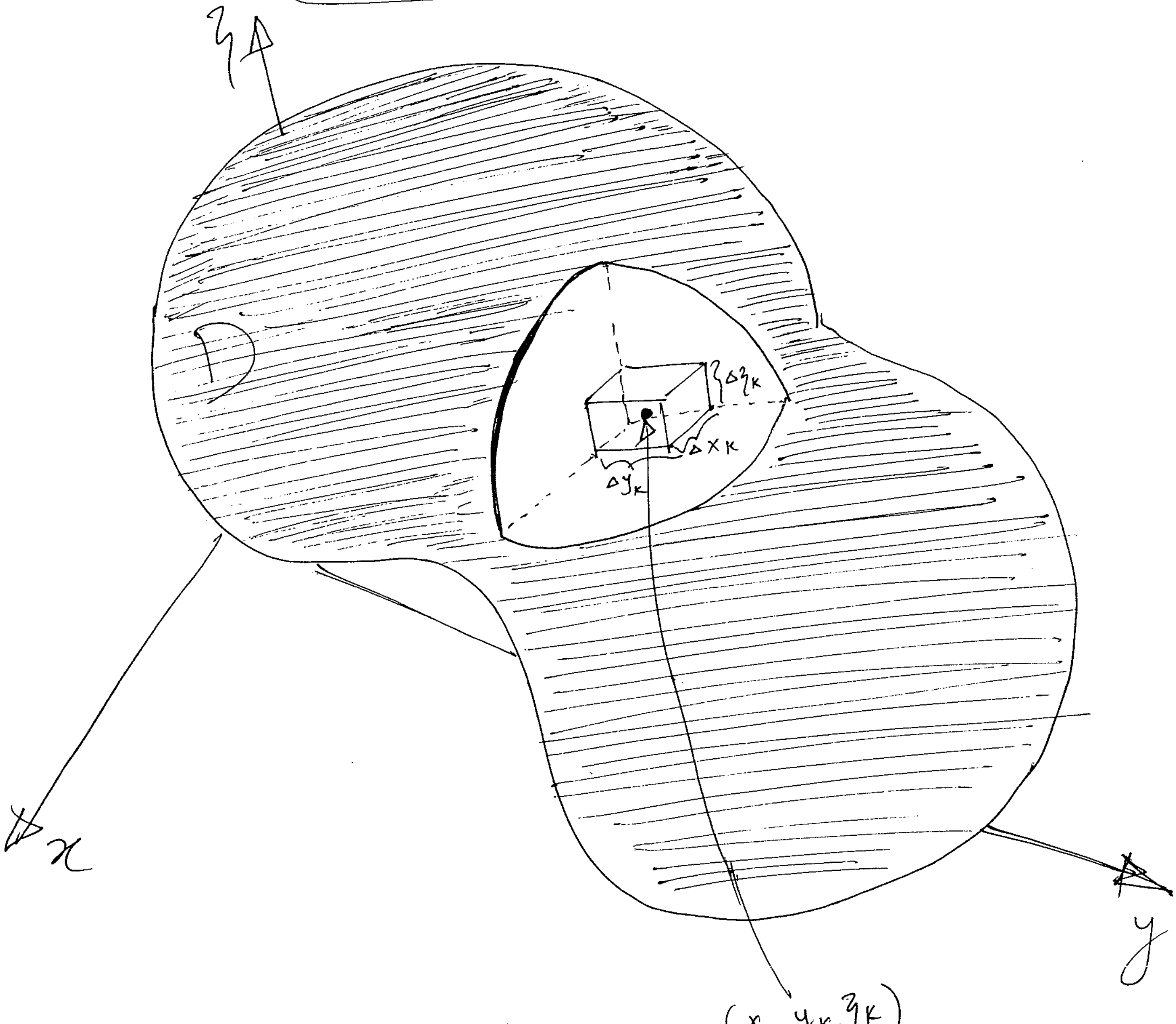
$$\boxed{\begin{aligned} r &= \rho \sin\phi ; \quad x = r\cos\theta = \rho \sin\phi \cos\theta \\ z &= \rho \cos\phi ; \quad y = r\sin\theta = \rho\sin\phi\sin\theta \\ \rho &= \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2} \end{aligned}}$$

(a) $\rho = a$ describes a sphere centered @ origin

(b) $\phi = \phi_0$ describes a single cone whose vertex lies @ origin & axis along z-axis

$\phi > \pi/2$, the cone opens downward.

§(13·4) Triple Integrals in Rectangular Co-ordinates



If $F(x,y,z)$ is a function defined on a closed bdd. region D in space — the region occupied by a solid ball, for eg. or a lump of clay — then the integral of F over D may defined in the following way. We partition a rectangular region containing D into rectangular cells by planes // to co-ordinate planes.

As before, $S_n := \sum_{K=1}^{n} F(x_K, y_K, z_K)\Delta V_K$

$$\lim_{n \to \infty} S_n = \iiint_D F(x,y,z)\,dV$$

if (i) F is continuous

(ii) D (bounding surface) is made of smooth surfaces joined along continuous curves.

This limit may also exist for some discontinuous f^ns.

Properties of Triple Integrals

If F & G are continuous, then

① $\iiint_D kF\,dV = k\iiint_D F\,dV$ (k const.)

② $\iiint_D (F \pm G)\,dv = \iiint_D F\,dV \pm \iiint_D G\,dV$

③ $\iiint_D F\,dV \geq 0 \Leftarrow F \geq 0$ on D

④ $\iiint_D F\,dV \geq \iiint_D G\,dV \Leftarrow F \geq G$ on D.

Defn :-

The Volm. of a closed, Bdd region D in space is

$$V = \iiint_D dV$$

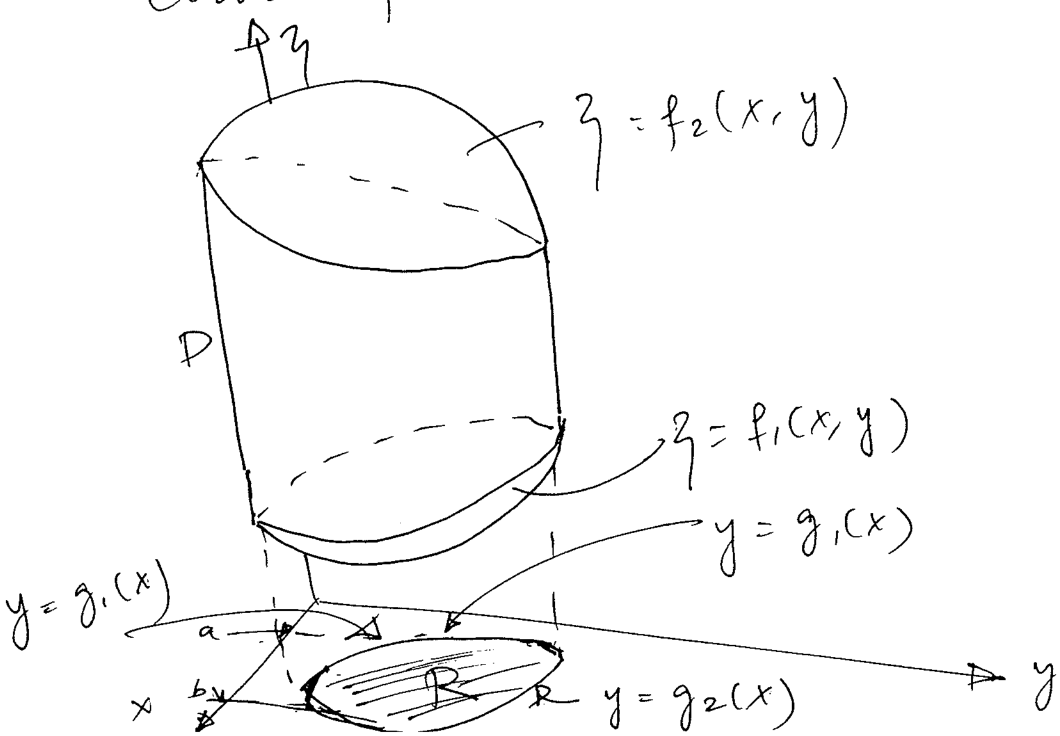
How to find limits of integration in Triple Integrals

$$\iiint_D F(x,y,z)\,dV$$

1st dz
2nd dy
3rd dx

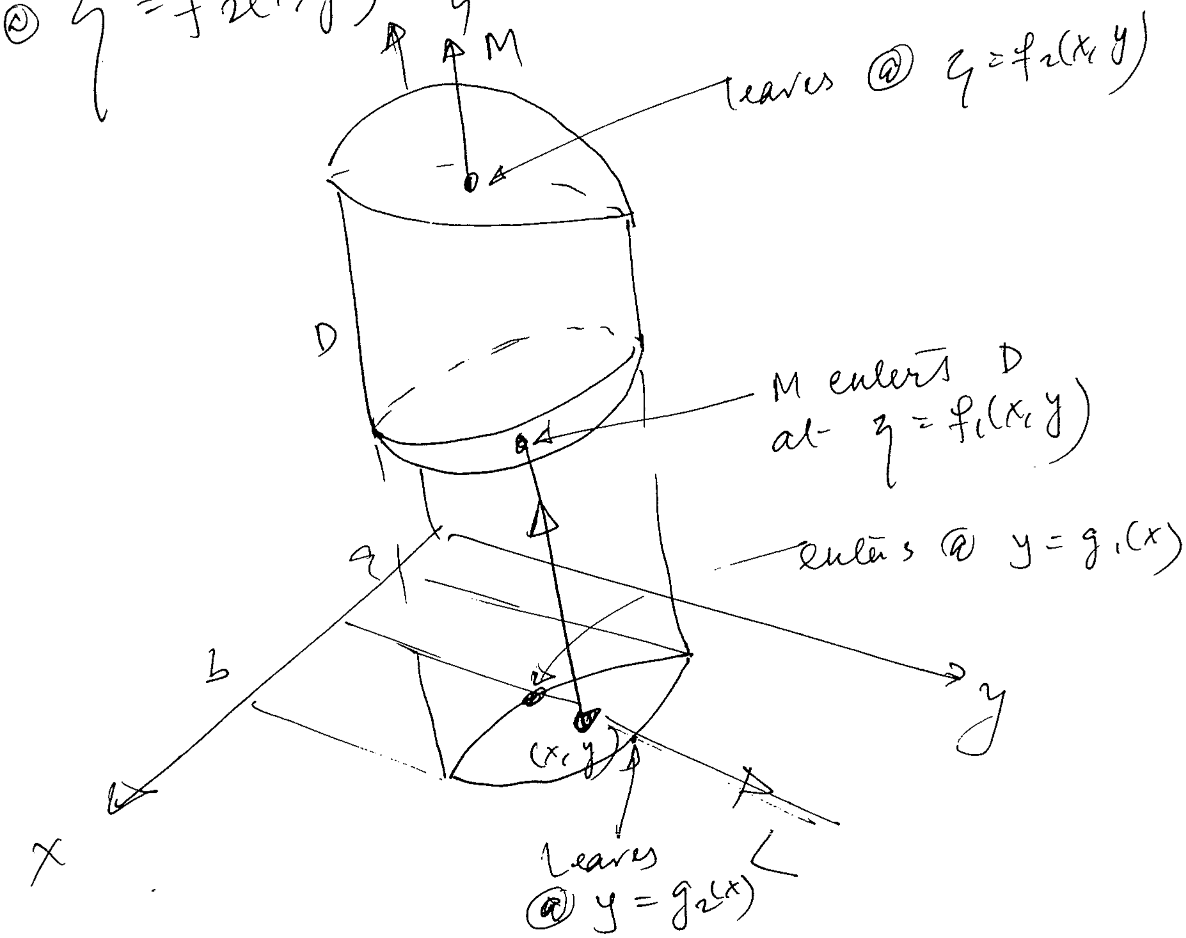
1) Sketch (a) 3-dim. region D and its shadow R on the xy-plane

(b) Label the upper & lower bounding surfaces of D and the upper & lower bounding curves of R.



(2) <u>Find the z-limits of integration.</u>

Shoot a lazer M from some pt (x, y) in the shadow region R $/\!/^{el}$ to z-axis. M enters D at $z = f_1(x,y)$ and leaves D @ $z = f_2(x,y)$



(3) <u>y-limits of integⁿ</u> :- Shoot another lazer L through (x,y) // y-axis. L enters R at $y = g_1(x)$ and leaves @ $y = g_2(x)$

(4) <u>x-limits of integⁿ</u> :- choose x-limits that include all lines through R $/\!/_{b}$ to y axis ($x = a$ & $x = b$)

$$\int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\, dz\, dy\, dx$$

Follow similar procedures if you change order of integration

* The shadow region, R of D lies in the plane of the last 2 variables w.r.t. which the iterated integ^n takes place.

eg: Find the vol^m of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$

Ans:- $V = \iiint\limits_D dz\,dy\,dx$

① $z = x^2 + 3y^2$ is the bottom bounding surface with elliptic c.s. (gradually inc. in size.
$z = 8 - x^2 - y^2$ is the upper bding surface with gradually dec. circular c.s.

They intersect in the curve

$$z = x^2 + 3y^2 = 8 - x^2 - y^2$$

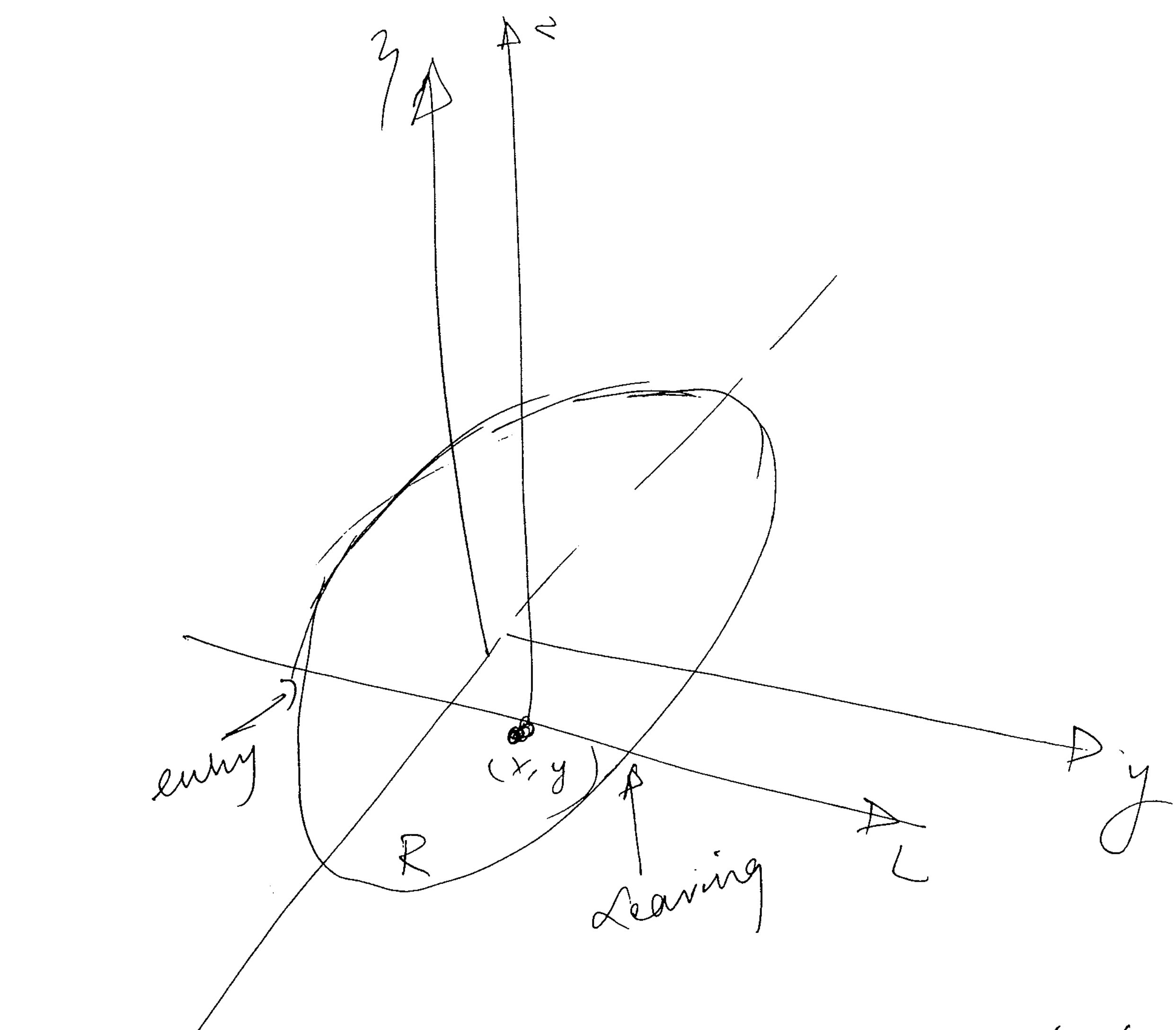
$$\Rightarrow 2x^2 + 4y^2 = 8$$

$$x^2 + 2y^2 = 4$$

$$\frac{x^2}{4} + \frac{y^2}{2} = 1 \quad \text{which is an ellipse.}$$

∴ The shadow of D on xy-plane is an ellipse $x^2 + 2y^2 = 4$

$$\Rightarrow y = \pm\sqrt{\frac{4 - x^2}{2}}$$



② Shoot M // z-axis through (x,y)

M enters D @ $z = x^2 + 3y^2$

& leaves D @ $z = 8 - x^2 - y^2$

③ Shoot L // y → enters R @ $y = -\sqrt{\frac{4-x^2}{2}}$

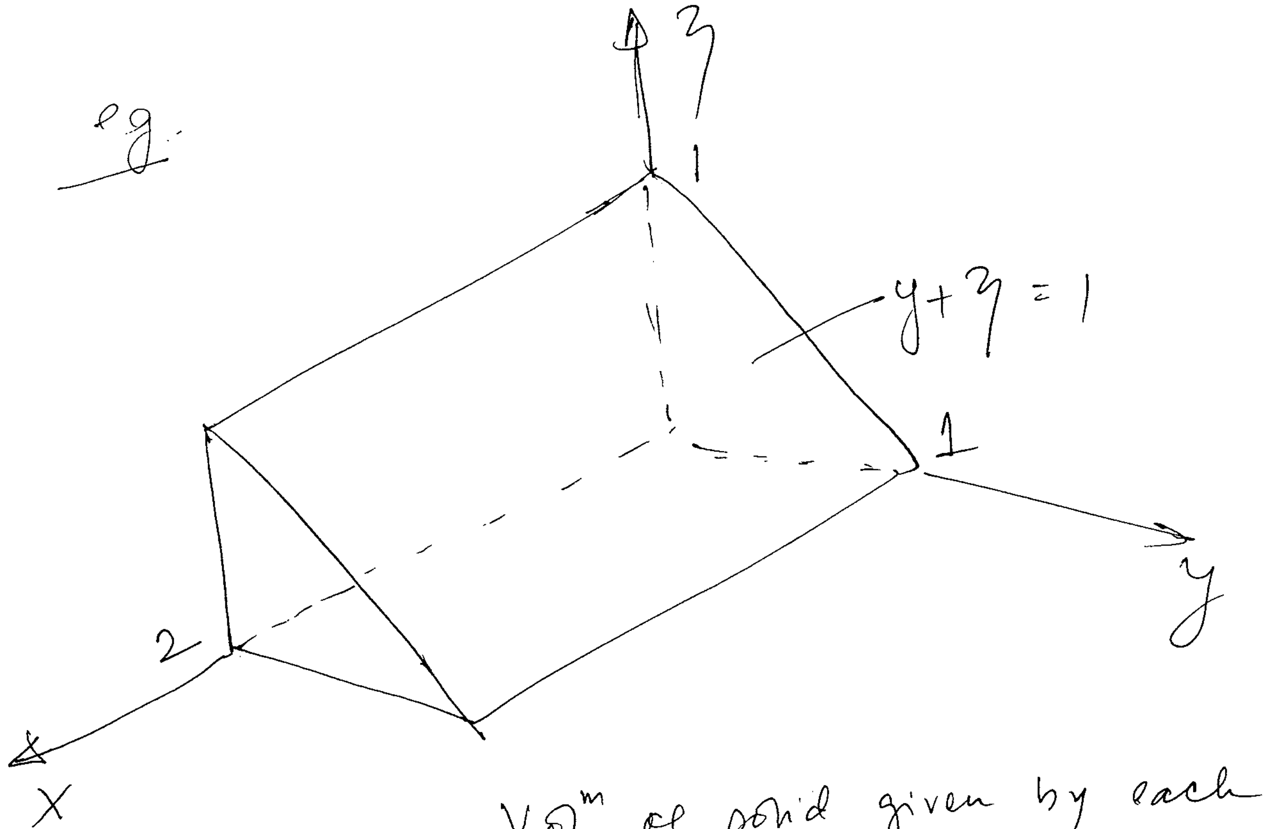
leaves R @ $y = \sqrt{\frac{4-x^2}{2}}$

④ $a^2 = 4 \Rightarrow a = \pm 2$

X-limits of integ^n $x = -2$ to $x = 2$

$$\therefore V = \int_{x=-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx = \ldots = 8\pi\sqrt{2}$$

eg.



Show :- Vol^m of solid given by each

a) $\int_0^1 \int_0^{1-z} \int_0^2 dx\,dy\,dz$

b) $\int_0^1 \int_0^{1-y} \int_0^2 dx\,dz\,dy$

c) $\int_0^1 \int_0^2 \int_0^{1-z} dy\,dx\,dz$

d) $\int_0^2 \int_0^1 \int_0^{1-z} dy\,dz\,dx$

e) $\int_0^1 \int_0^2 \int_0^{1-y} dz\,dx\,dy$

f) $\int_0^2 \int_0^1 \int_0^{1-y} dz\,dy\,dx$.

Avg. Value of a F^n in Space

Avg. value of F over region D in space, $:= \dfrac{1}{\text{vol}^n(D)} \iiint_D F \, dV$.

7/24/2010 § (13.6) Triple Integrals in Cylindrical and Spherical Co-ordinates

Cylindrical Co-ordinates

Volume Element :-

$$dV = r\,dr\,d\theta\,dz$$

How to integrate in Cylindrical Co-ordinates

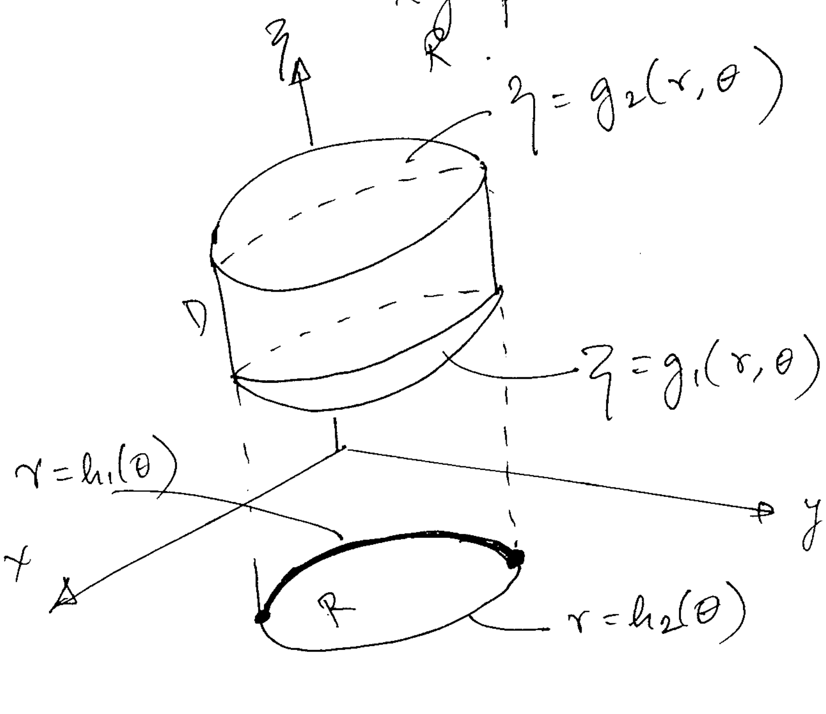
To evaluate :- $\iiint_D f(r,\theta,z)\,dV$

D ⟶ is in Cylindrical Co-ordinates

order of integration ⟶ $dz, dr, d\theta$

Steps :-

① Sketch :- D and its shadow R on the xy plane. label the bounds of D & R.

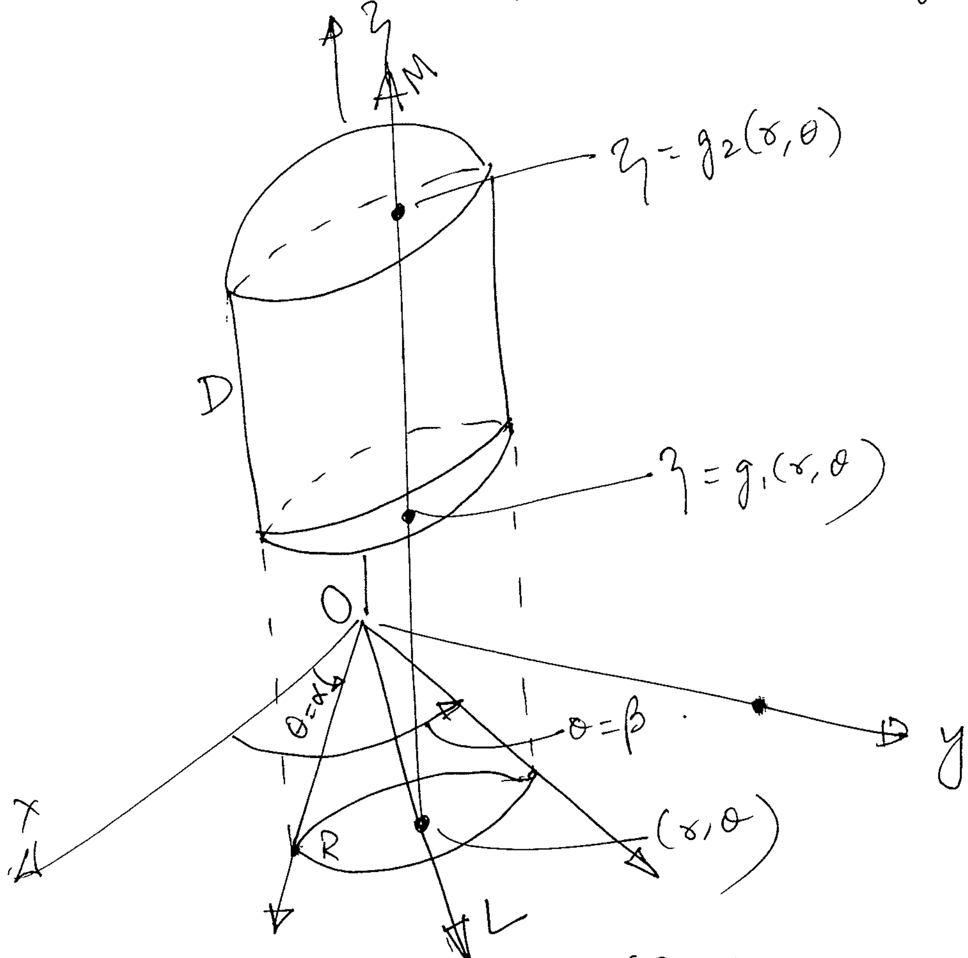


② z- limits of integration

Shoot a laser, M through (r,θ) in R // z-axis

M enters D @ $z = g_1(r,\theta)$ & leaves D @ $z = g_2(r,\theta)$

& these are the z-limits of integration.



③ r- limits of integration

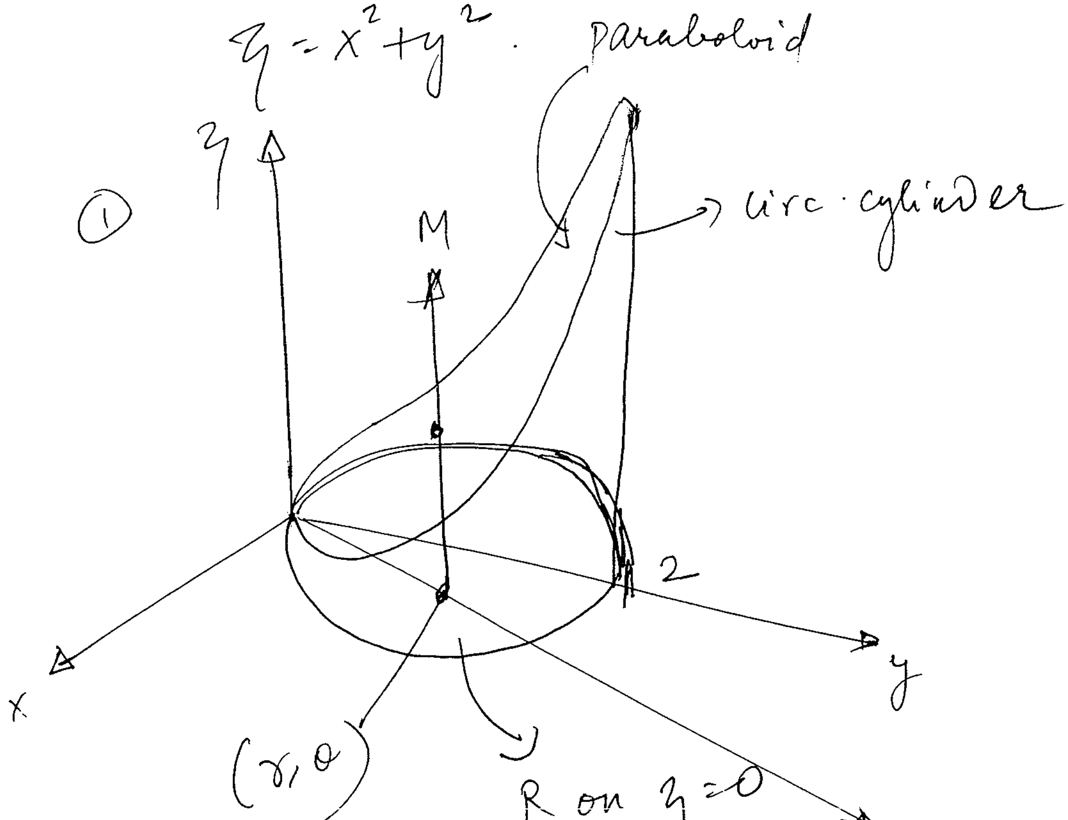
Shoot a lazer L from O(0,0,0) through (r,θ). L enters R @ $r = h_1(\theta)$ and leaves @ $r = h_2(\theta)$; which are the r-limits of integration.

④ θ - limits of integration :- Sweep the lazer L across R to find the bounds $\theta = \alpha$ and $\theta = \beta$ with the +ve x-axis.

$$\therefore \int_{\theta=\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r,\theta,z)\,dz\,(r\,dr)\,d\theta .$$

eg: Find the limits of integration in cylindrical co-ordinates for integrating a f^n $f(r, \theta, z)$ over the region D bounded below by the plane $z=0$, laterally by the circular cylinder $x^2+(y-1)^2=1$ and above by the paraboloid

$z = x^2+y^2$.



R on $z=0$
is a circle
$x^2+(y-1)^2=1$
centered at
$(0,1)$

or equivalently

$$x^2+y^2+1-2y=1$$

$$\Rightarrow r^2 - 2r\sin\theta = 0$$

$$\Rightarrow \boxed{r = 2\sin\theta}$$

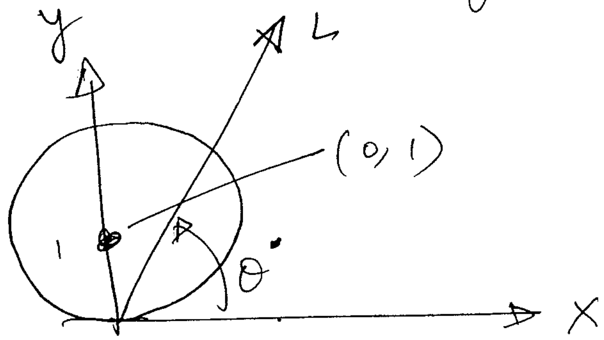
(ii) z - limits of integration

M enters D @ $z = 0$
& leaves " " $z = x^2 + y^2 = r^2$

(iii) r - limits of integration

L enters R @ $r = 0$
& leaves R @ $r = 2\sin\theta$

(iv) θ - limits of integration



$\theta = 0$ to π

$$\therefore \int_0^{\pi} \int_0^{2\sin\theta} \int_0^{r^2} f(r, \theta, z)\, dz\, (dr)\, d\theta$$

Spherical Co-ordinates

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

How to integrate in Spherical Co-ordinates

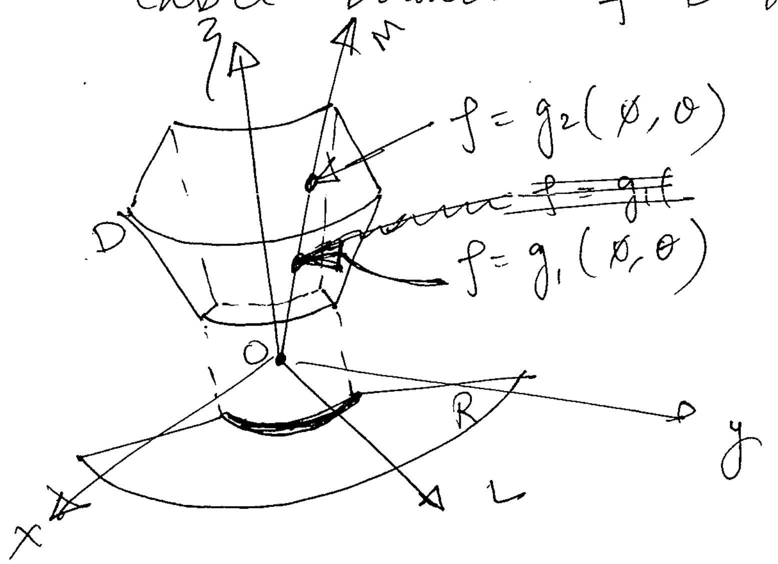
$$\iiint_D f(\rho, \phi, \theta) \, dV$$

D in Spherical Co-ordinates

Order of integration :- $d\rho$, $d\phi$, $d\theta$

Steps :-

① **Sketch** D & its shadow R on xy-plane
Label bounds of D and R.



② ρ - limits :- Shoot a Lazer M from $O(0,0,0)$ at an $\angle^{le}$ ϕ with +ve z-axis
Draw projection of M on xy-plane, call it L.
L makes an $\angle^{le}$ θ with +ve x-axis. M leaves D @ $\rho = g_2(\phi, \theta)$ & enters D @ $\rho = g_1(\phi, \theta)$.

(3) ϕ - limits :- For any given θ, the $\angle^{le}$ ϕ that M makes with the z-axis runs from $\phi = \phi_{min}$ ~~one~~ to $\phi = \phi_{max}$ which are the ϕ limits of integration.

(4) θ - limits :- The layer L sweeps over R as θ runs from α to β.

$$\therefore \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{min}}^{\phi=\phi_{max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho,\phi,\theta)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

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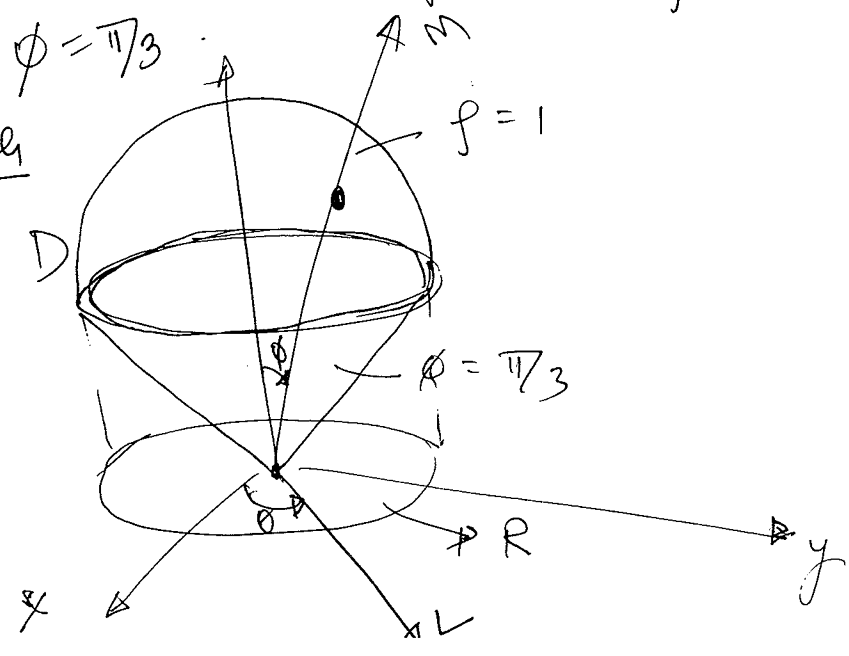
eg Find ~~the~~ volume of the upper region D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \pi/3$.

Ans:- (1) Sketch



② ρ- limits :-

M enters D @ $\rho = 0$ & leaves D @ $\rho = 1$

③ ϕ - limits

$\theta = 0$ to $\pi/3$ ($\because > \pi/3$ is outside cone $\theta = \pi/3$)

④ θ - limits :-

$\theta = 0$ to 2π

$$\therefore V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \left(\frac{\rho^3}{3}\right)_0^1 \sin\phi \, d\phi \, d\theta$$

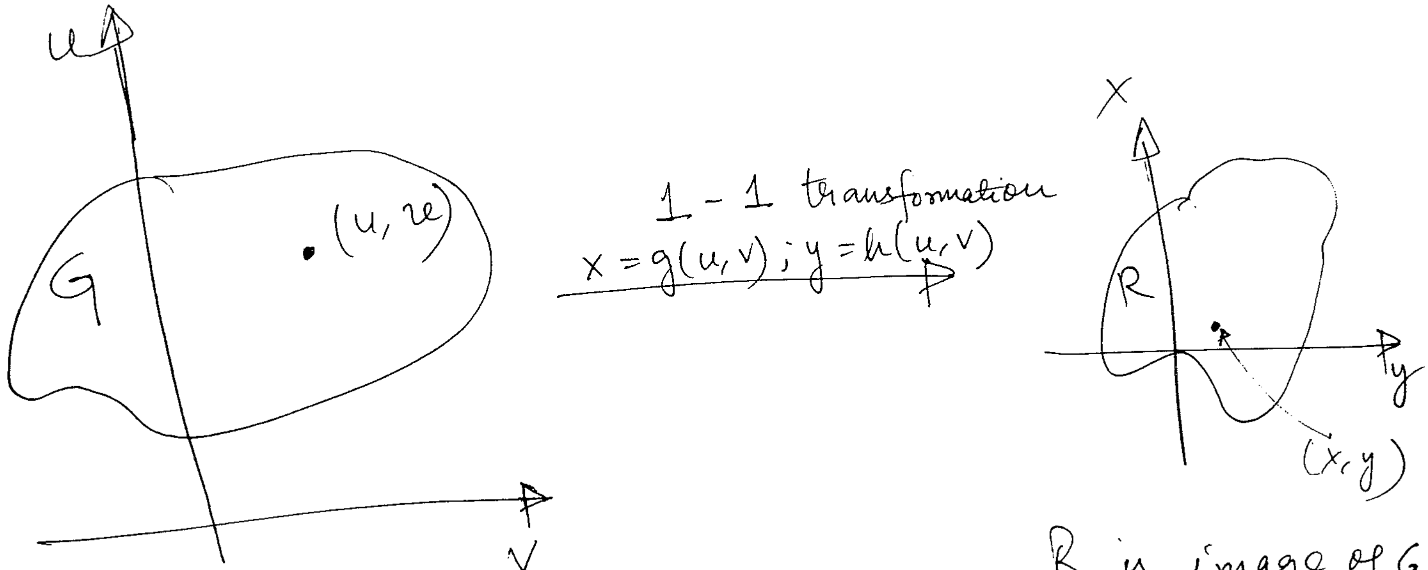
$$= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \left(-\cos\theta\right)_0^{\pi/3} d\theta = \int_0^{2\pi}\left(-\frac{1}{6}+\frac{1}{3}\right) d\theta$$

$$= \frac{1}{6}(2\pi)$$

$$= \pi/3.$$

§(13.7) Co-ordinate Transformation



R is image of G

G is preimage of R

$$f(x, y) \equiv f\Big(g(u,v),\, h(u,v)\Big)$$

Ques :- How is $\int_R f(x,y)\, dA$ related to

$$\int_G f\big(g(u,v),\, h(u,v)\big)\, d\tilde{A}$$

Ans :- $$\iint_R f(x,y)\, dx\, dy = \iint_G f\big(g(u,v),\, h(u,v)\big)\, |J(u,v)|\, du\, dv \quad \text{(1)}$$

absolute value of $J(u,v)$

matrix determinant

where $J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ where $x = g(u,v)$, $y = h(u,v)$

Q) When does above (eqn ①) exist?

Ans) (a) If g, h, and f have continuous partials

(b) $J(u, v)$ is 0 only at isolated pts.

Notation :-

$$J(u, v) = \frac{\partial(x, y)}{\partial(u, v)}$$

Ques) For polar co-ordinates

$(r - \theta) \equiv (u - v)$

$x = r\cos\theta$
$y = r\sin\theta$

$$J(r, \theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

$$\therefore \iint_R f(x, y)\, dx\, dy = \iint_G f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

exercise

Show the 3-d X^n from $(x, y, z) \to (\rho, \phi, \theta)$

$$J(\rho, \phi, \theta) = \rho^2 \sin\phi$$