Lecture (5) :-

23/1/19

(5.1) Cauchy - Riemann Conditions & Analyticity

Let $f(z) = u(x,y) + i v(x,y)$

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \qquad \text{--- (1)}$$

Since the complex plane spans the 2D Euclidean plane, for the above limit to exist, the R.H.S must evaluate to the same limiting value irrespective of the path chosen.

Let us consider two different paths.

(a) $\Delta z = \Delta x$ (along or $\parallel^{el}$ to x-axis).

$$f'(z) = \lim_{\Delta x \to 0} \frac{f(z+\Delta x) - f(z)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y) + i\{v(x+\Delta x, y) - v(x,y)\}}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \qquad \text{--- (2)}$$

(b) $\Delta z = i \Delta y$ (along or $\parallel^{el}$ to img. axis).

$$f'(z) = \lim_{i\Delta y \to 0} \frac{(f(z+i\Delta y) - f(z))}{i \Delta y}$$

$$= \lim_{i\Delta y \to 0} \frac{u(x, y+i\Delta y) - u(x,y) + i\{v(x, y+i\Delta y) - v(x,y)\}}{i\Delta y}$$

$$= \lim_{\Delta y \to 0} (-i) \left\{\frac{u(x,y+i\Delta y) - u(x,y)}{\Delta y}\right\} + \left\{\frac{v(x,y+i\Delta y) - v(x,y)}{\Delta y}\right\}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{---} \quad (3)$$

Comparing eqns (2) & (3) (i.e. cmp real & img parts)

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}\right\} \quad \text{---} \quad (4)$$

these eqns. are called the Cauchy-Riemann (CR) conditions.

<u>Theorem</u> (Differentiability of complex no.s).

$f(z) = u(x,y) + iv(x,y)$ is differentiable at a pt. $z = x+iy$ of a region in the complex plane.

$\Longleftrightarrow$ the partial derivatives u_x, u_y, v_x, v_y exist & are continuous & satisfy the CR eqns. (4) at $z = x+iy$.

This means if and only if (iff)

<u>Note</u> :- the CR eqns by themselves are necessary conditions for differentiability

(5.2) Orthogonality of level curves :- $u(x,y)=c_1 \perp v(x,y)=c_2$.

$\because$ u and v are f^n's of x & y.

$$\nabla u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

$$\nabla v = \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

$$\nabla u \cdot \nabla v = \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$

if CR conditions are satisfied

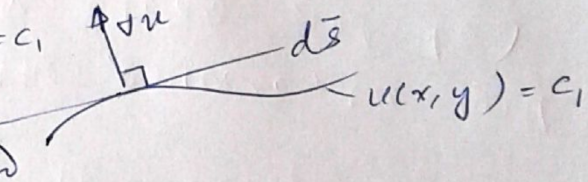
$$\frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0.$$

$$\Rightarrow \nabla u \perp \nabla v \quad \text{——— (1)}$$

Recall the definition of directional derivative

$$du = \nabla u \cdot \underbrace{d\vec{s}} \quad \text{——— (2)}$$

$\rightarrow$ tangent to the curve $u(x,y)=c_1$



$\because$ (1) is true if CR conditions are satisfied

then (2) $\Rightarrow$ $u(x,y)=c_1 \perp v(x,y)=c_2$

this means for every analytic (differentiable) f^n $f(z) = u(x,y) + i v(x,y)$ for which the CR conditions are true (b/c of th^m in sec. 5.1); the level curves formed by the real & imaginary parts of $f(z)$ are perpendicular to each other.

Defn :- (Analyticity)

A f^n $f(z)$ is said to be <u>analytic</u> at a point z_0 if $f(z)$ is <u>differentiable</u> in a neighborhood of z_0.

* $f(z)$ is analytic in a region if it is analytic at every pt. in the region.

* "Analytic" $\equiv$ "Holomorphic".

* Later on in the course, we will learn about another related (but <u>not</u> equivalent) term called "Meromorphic".

<u>Entire f^n</u> :- A f^n that is analytic at every pt. in the "entire" <u>finite</u> plane.

eg e^z, $\cos z$, $\sin z$, z^n, etc ...

eg: <u>(Application of CR conditions)</u> :

$\bar{z}$ is <u>not analytic</u> anywhere in $\mathbb{C}$.

Why?. B/c $f(z) = \bar{z} = x - iy$
$\equiv u - iv$

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$\frac{\partial u}{\partial x} = 1$; $\frac{\partial v}{\partial y} = -1$

$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \Rightarrow$ CR condn's fail $\Rightarrow$ $f(z) = \bar{z}$ is not diff. anywhere

(5.3) Analyticity vis-à-vis Laplace eqn.

Let $f(z) = u(x,y) + i v(x,y)$ be analytic

$$\therefore \nabla^2 u = \partial_x^2 u + \partial_y^2 u$$

$$\overset{\text{CR}}{\underset{\text{cond}^n}{=}} \partial_x\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \partial_x \partial_y v - \partial_y \partial_x v$$

$$\underset{\text{are cont.}}{\overset{v \text{ \& its derivatives}}{=}} \partial_x \partial_y v - \partial_x \partial_y v \quad \text{by Fubini's th}^m.$$

$$= 0.$$

Likewise $\nabla^2 v = 0$.

And hence we say $u(x,y)$ & $v(x,y)$ are harmonic f^ns & v is called the harmonic conjugate of u.

<u>Def^n</u> (Harmonic f^n) :- Any f^n $\omega(x,y)$ satisfying $\nabla^2\omega = 0$ (i.e. Laplace eq^n) in a domain D is called an harmonic f^n in D.

<u>Theorem</u> :- $f(z) = u(x,y) + i v(x,y)$, is an analytic f^n iff u & v satisfy the Laplace's eqns & v is the harmonic conjugate of u. <u>Pg15</u>)

Physical application :- Ideal fluid flow (2D).

$\rightarrow$ Laplace eqn

$\rightarrow$ Complex variable technique.

Ideal flow (i) $\partial_z = 0$

ii) incompressible/Solenoidal $\nabla \cdot \bar{v} = 0$

iii) Steady state i.e. $\partial_t \bar{v} = 0$

$\exists\, \varphi$ s.t. $\bar{v} = \nabla\varphi \Leftarrow$ iv) Irrotational :- $\nabla \times \bar{v} = 0$ (locally).

ii) $\Rightarrow$ $\partial_x v_1 + \partial_y v_2 = 0$; $\vec{v} = (v_1, v_2)$ — (a)

iv) $\Rightarrow$ $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ v_1 & v_2 & 0 \end{vmatrix} = \partial_x v_2 - \partial_y v_1 = 0$ ———— (b)

The basic idea is to find an analytic f^n $f = \varphi + i\psi$ that satisfies eqn. (a) & (b).

Analyticity of $f \Rightarrow$ $\left.\begin{array}{l} \dfrac{\partial \varphi}{\partial x} = \dfrac{\partial \psi}{\partial y} \;\&\; \dfrac{\partial \varphi}{\partial y} = -\dfrac{\partial \psi}{\partial x} \end{array}\right\}$ — (c) CR eqns.

$\frac{\partial \varphi}{\partial x} = v_1$, $\frac{\partial \varphi}{\partial y} = v_2$ (b/c $v = \nabla\varphi$)

Now substituting (c) in (a) & (b) gives.

$\nabla^2 \varphi = 0$ & $\nabla^2 \psi = 0$ ———— (d)

Here φ is the velocity potential & ψ is stream f^n.

Solving (d) gives us the required potential f^n

$$f(z) = \varphi(x,y) + i\psi(x,y)$$

$$\text{s.t. } f'(z) = \frac{\partial \varphi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$\underset{\text{eqns}}{\overset{CR}{=}} \frac{\partial \psi}{\partial y} - i\frac{\partial \varphi}{\partial y}$$

$$\overset{\text{or}}{=} \frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y}$$

$$= v_1 - iv_2$$

$$\Rightarrow \overline{f'(z)} = v_1 + iv_2 \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \vec{v}$$

Thus the complex conjugate of the potential f^n gives us all information abt. the flow (i.e. info about $\vec{v}$)

It follows that $\underbrace{\psi(x,y) = c_1}_{\text{streamline of flow}} \perp \varphi(x,y) = c_2$

$$\because \underset{\vec{v}}{\underset{\parallel}{\nabla\varphi}} \perp \varphi(x,y) = c_2$$

$\Rightarrow \vec{v}$ is in the dir^n. of $\psi = c_1$

or ψ is in the direction of the flow field $\vec{v}$.

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