PLAKSHA

Engineering Math in Action: FM 103
Quiz-2 for section-L1

Total time: 30 mins
December 07, 2023

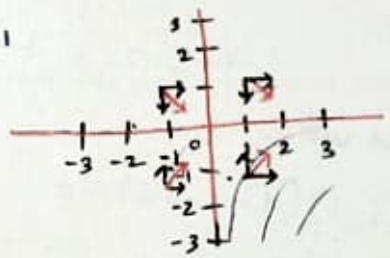
Full Name: _____ **UID:** _____

Instructions: You must **not** be in possession of any cheat sheet, notes, or electronic devices like laptops or calculators inside the examination hall. Show **all steps** leading to your final answer to receive any credit for your solution. Merely stating the final answer may not fetch you any credit. The maximum score allotted to each question is mentioned in the square bracket on the right margin. **Maximum score for this quiz is 15.**

==========START OF QUESTIONS==========

1. Draw the direction field for the differential equation $y' = -y$. [4]
Solution:- Rewriting $\frac{dy}{dt} = -y$, $\frac{dx}{dt} = 1$

x	y	dy/dt	dx/dt
1	1	-ve	+ve
-1	1	-ve	+ve
-1	-1	+ve	+ve
1	-1	+ve	+ve



2. Write a second-order linear differential equation with constant coefficients using mathematical notations. [3]
Solution:-
$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2 y = 0$$

3. Find the solution to the system of linear differential equations
$$x' = -y$$
$$y' = x$$
with the initial conditions $x(0) = 1$ and $y(0) = 0$. [4]
Solution:-

$$\frac{dX}{dt} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} X, \quad \dot{X}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ with } X = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0.$$

$$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm i$$

for $\lambda = i$

$$(A - \lambda I)u = 0$$

$$\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0$$

$$-iu_1 - u_2 = 0$$
$$u_1 - iu_2 = 0.$$
$$-u_2 = iu_1$$
$$u_1 = i, \; u_2 = 1$$

eigen vector $\begin{bmatrix} i \\ 1 \end{bmatrix}$

for $\lambda = -i$

$$(A - \lambda I)u = 0.$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = 0.$$

$$iu_1 - u_2 = 0.$$
$$u_1 + iu_2 = 0.$$
$$iu_1 = u_2.$$
$$u_1 = -i, \; u_2 = +1$$

eigen vector $\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Gen soln:
$$X = c_1 e^{-it}\begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 e^{it}\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

$$X = c_1(\cos t - i\sin t)\begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2(\cos t + i\sin t)\begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$X = c_1\begin{bmatrix} -i\cos t - \sin t \\ \cos t - i\sin t \end{bmatrix} + c_2\begin{bmatrix} i\cos t - \sin t \\ \cos t + i\sin t \end{bmatrix}$$

$$X = \begin{bmatrix} -c_1\sin t - c_2\sin t \\ c_1\cos t + c_2\cos t \end{bmatrix} + i\begin{bmatrix} -c_1\cos t + c_2\cos t \\ -c_1\sin t + c_2\sin t \end{bmatrix}$$

$$X = 0(c_1 + c_2)\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + (-c_1 + c_2)i\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

$$X = C_1\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} + C_2\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

4. Solve the ODE $y''' - 4y'' + 5y' - 2y = 0$ with the initial conditions $y(0) = 3$, $y'(0) = 4$ and $y''(0) = 7$. [4]

Solution:-

Auxiliary eqn will be.

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0.$$

Put $\lambda = 1$

$$1 - 4 + 5 - 2 = 1 - 1 = 0.$$

So one root will be $\lambda - 1$.

$$(\lambda - 1)(\lambda^2 - 3\lambda + 2) = 0$$

$$(\lambda - 1)^2 (\lambda - 2) = 0$$

$$\lambda = 1, 1, 2.$$

Gen. solution $y(t) = c_1 e^t + c_2 t e^t + c_3 e^{2t}$.

Putting $y(0)$

$$3 = c_1 + c_3.$$

$$y'(t) = c_1 e^t + c_2 e^t + c_2 t e^t + 2c_3 e^{2t}.$$

$$y'(0) = 4.$$

$$4 = c_1 + c_2 + 2c_3$$

$$y''(t) = c_1 e^t + c_2 e^t + c_2 e^t + c_2 t e^t + 4c_3 e^{2t}.$$

$$y''(0) = 7.$$

$$7 = c_1 + 2c_2 + 4c_3$$

$$8 = 2c_1 + 2c_2 + 4c_3$$

$$7 = c_1 + 2c_2 + 4c_3$$

$$c_1 = 1, \quad c_3 = 2, \quad c_2 = -1.$$

$$y(t) = e^t - t e^t + 2e^{2t} = e^t(-t + 2e^t + 1)$$

Hints:

Question 1

- Consider ordered (paired) coordinates (y, t) in all four quadrants.
- Compute and tabulate slopes at each of the above coordinates.
- Draw the slopes as short line segments to populate the slope field.

Question 2

- Make sure to represent each term of your example using standard mathematical notation.

Question 3

- Write the system of equations in matrix-vector form.
- Compute the eigenvalues and eigenvectors of the coefficient matrix.
- Write the general solution by using the principle of superposition.
- Apply the initial conditions and write the full solution for the given system.

Question 4

- Write the characteristic equation for the given problem.
- Find the roots of the characteristic polynomial.
- Investigate the nature of the above roots and accordingly write the general form of the full solution by applying the principle of superposition.
- Apply the initial conditions and write the full solution for the given differential equation.

Solution-3

$$X(0) = k \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_2 = 1., \quad c_1 = 0.$$

$$X = \begin{bmatrix} \text{Cost} \\ \text{Snt} \end{bmatrix}$$

$x = \text{Cost}, \quad y = \text{Snt}.$