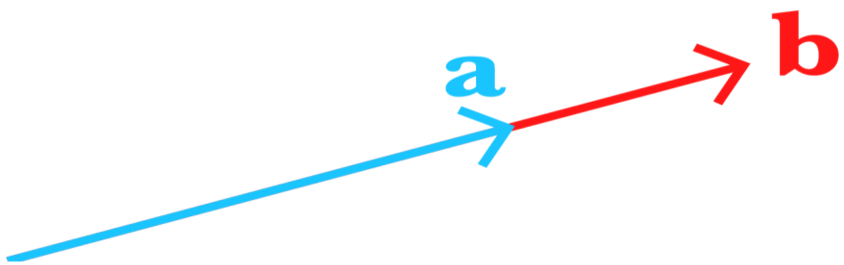
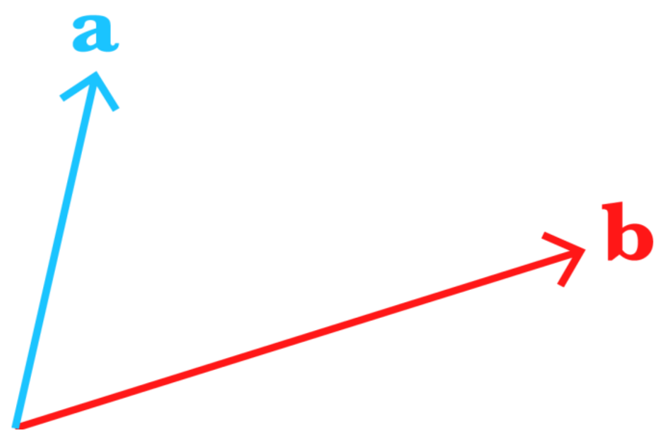
Linear Algebra

Engineering Mathematics In Action

Linear Dependence and Independence of two vectors



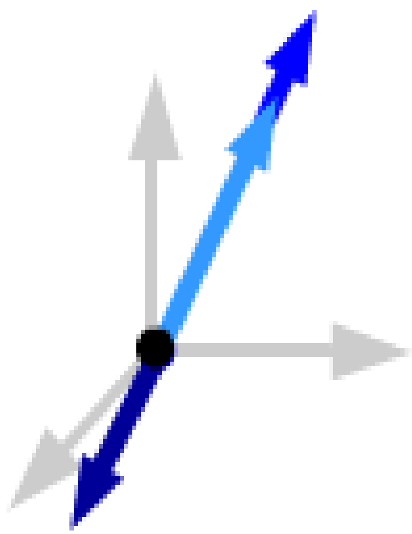


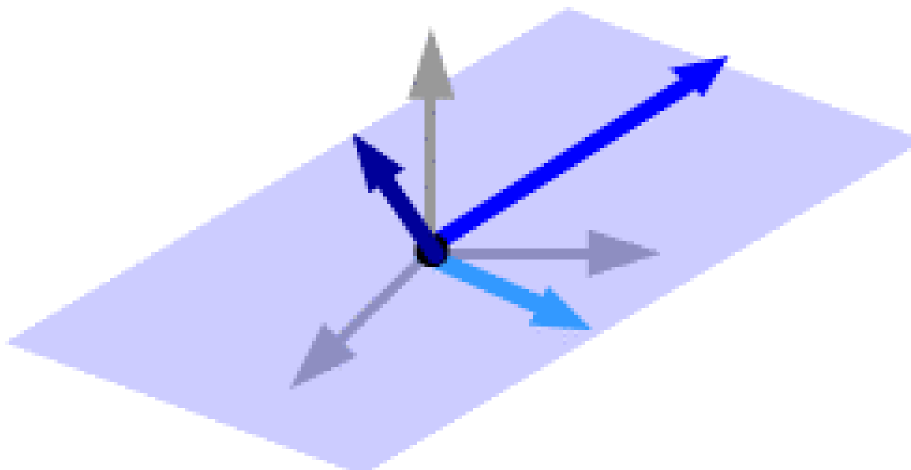
Dependent vectors: If the two vectors have same direction (parallel).

Independent vectors: If the two vectors have different direction (not parallel).

Linear Dependence and Independence of two vectors

This can be visualized in 3D as well.





Dependent vectors: If the two vectors have same direction (parallel).

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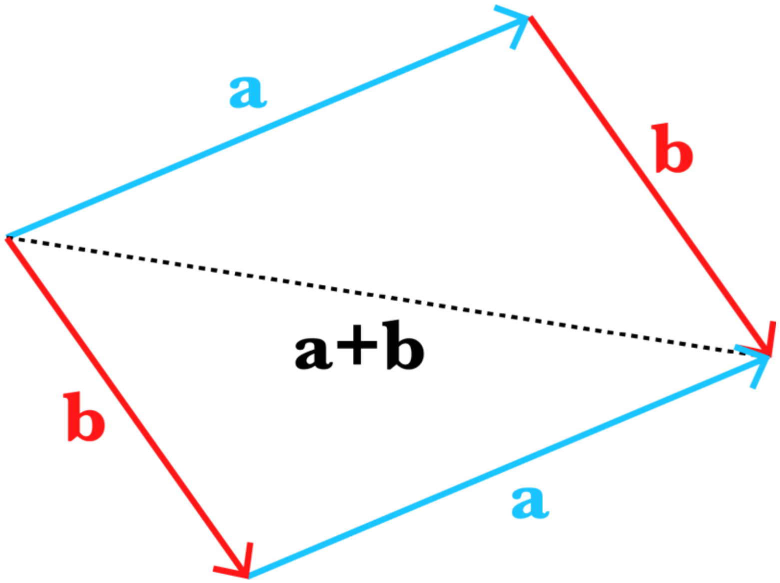
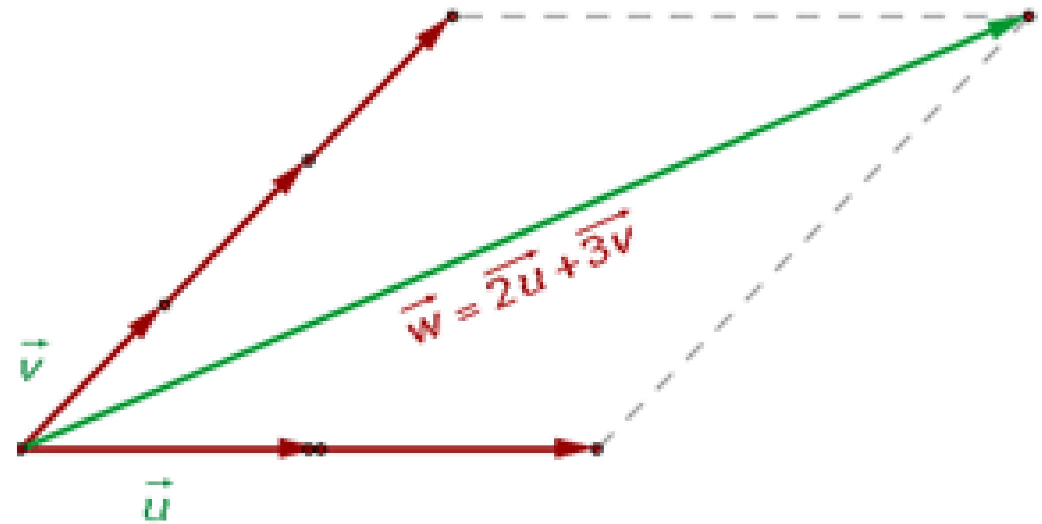
Linear Combinations of two vectors





Here, **a** and **b** are independent vectors but **a + b** vector is depending on vectors **a** and **b**.

Here, **u** and **v** are independent vectors but **u + v** vector is depending on vectors **u** and **v**.

We say that **a + b** is a linear combination of vectors **a** and **b** and **2u + 3v** is a linear combination of vectors **u** and **v**. Similarly, we can make a linear combination of **k** vectors.

Linear Dependence and Independence of k vectors

Definition (Linearly_dependent_vectors):
Let $\mathcal{V}$ be a vector space and $\mathcal{X} \subset \mathcal{V}$ be a non-empty subset. Then $\mathcal{X}$ is ***linearly dependent*** if there are distinct vectors $v_1, v_2, \ldots, v_k \in \mathcal{X}$, *and scalars* $c_1, c_2, \ldots, c_k$ ***(not all of them zero)***, *s.t.* $c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0$.

This is equivalent to saying that *at least one of the vectors v_i can be expressed as a linear combination of the others.*

$$v_i = \sum_{j \neq i} - \left(\frac{c_j}{c_i}\right) v_j$$

Definition (Linearly independent vectors):
A subset which is not linearly dependent is said to be ***linearly independent.***
Thus a set of distinct vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if an equation of the form $c_1 v_1 + c_2 v_2 + \ldots + c_k v_k = 0$ always implies that $c_1 = c_2 = \ldots = c_k = 0$.

Geometrical Interpretation of Linear Dependence

Let V_1, V_2, V_3 be the vectors in 3D-Euclidean space $\mathbb{R}^3$ with a common origin. If these vectors form a *linearly dependent* set, then one of them, say V_1, can be expressed as a linear combination of the other two: $V_1 = aV_2 + bV_3$. This implies, by the parallelogram law, that the three vectors are **co-planar.**

In fact, **linearly dependent set of vectors with common origin $\Leftrightarrow$ co-planar.**

Can you think of a similar interpretation of vectors in $\mathbb{R}^2$?

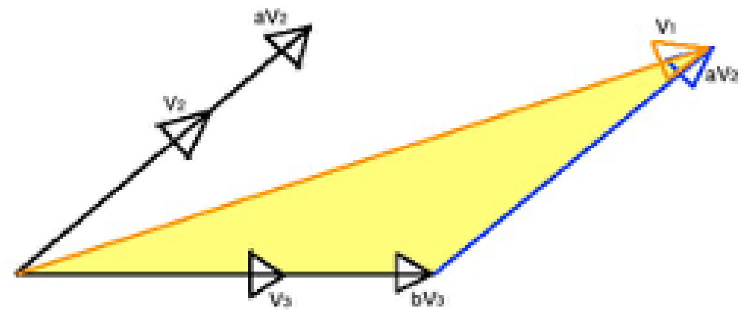
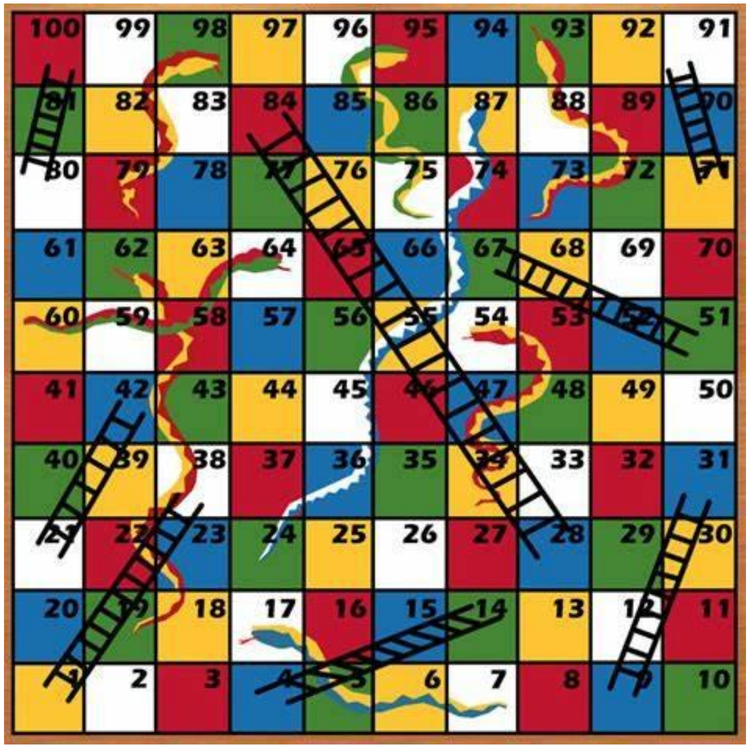


Fig. 1: Linear dependence of vectors is equivalent to coplanar geometry

Some Examples of Existence of Vectors



VECTORS, VECTORS, EVERYWHERE

Basis of a Vector Space

Given a vector space, basis is a smallest set of vectors whose linear combinations can give all vectors in the vector space.

In other words, basis is the collection of least number of vectors (directions) required to span the whole vector space.

Note: Least number of set refers to set of linearly independent vectors.

Spanning of set: collection of all possible linear combinations**.**

Basis of a Vector Space

Let $\mathbb{X}$ be a non-empty subset of a vector space $\mathcal{V}$. Then $\mathbb{X}$ is called a *basis* of $\mathcal{V}$ if **both** the following are true:

i. $\mathbb{X}$ is linearly independent

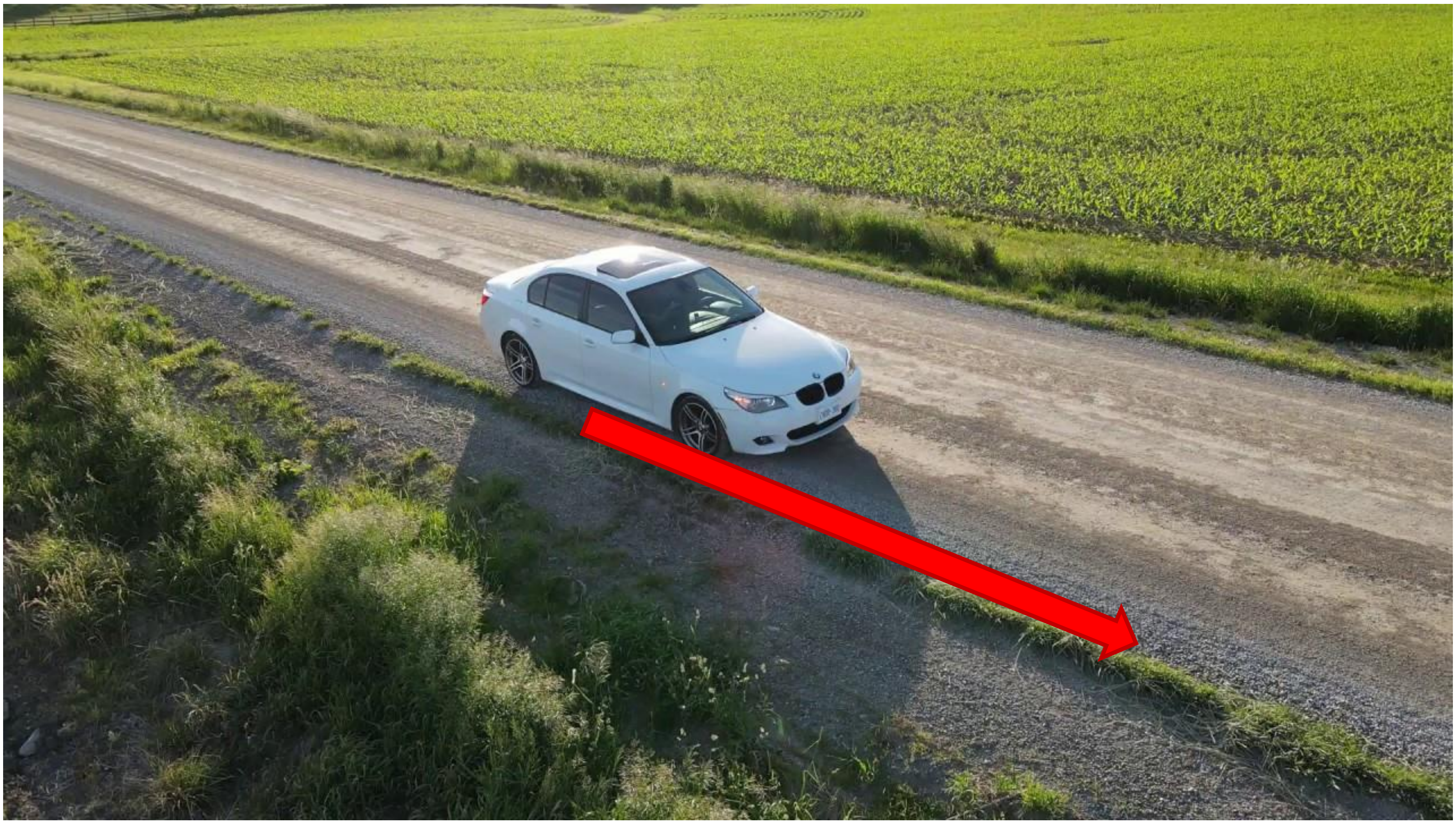
cannot generate an element of $\mathbb{X}$ as linear combination of the other elements of $\mathbb{X}$

ii. $\mathbb{X}$ generates $\mathcal{V}$ (i.e. $\mathbb{X}$ *spans* $\mathcal{V}$)

any element of $\mathcal{V}$ can be generated as a linear combination of the elements of $\mathbb{X}$

Basis of R

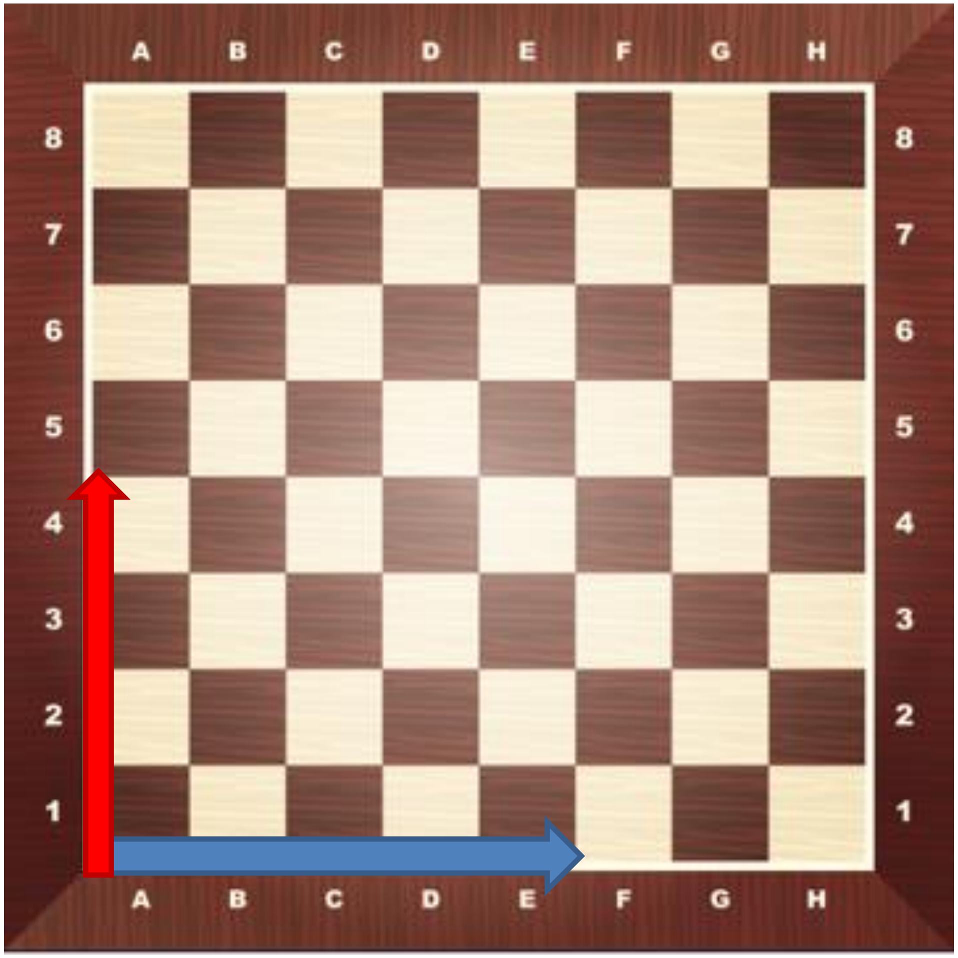
Standard Basis: {1}
Other examples: {5}, {0.3}, {π}

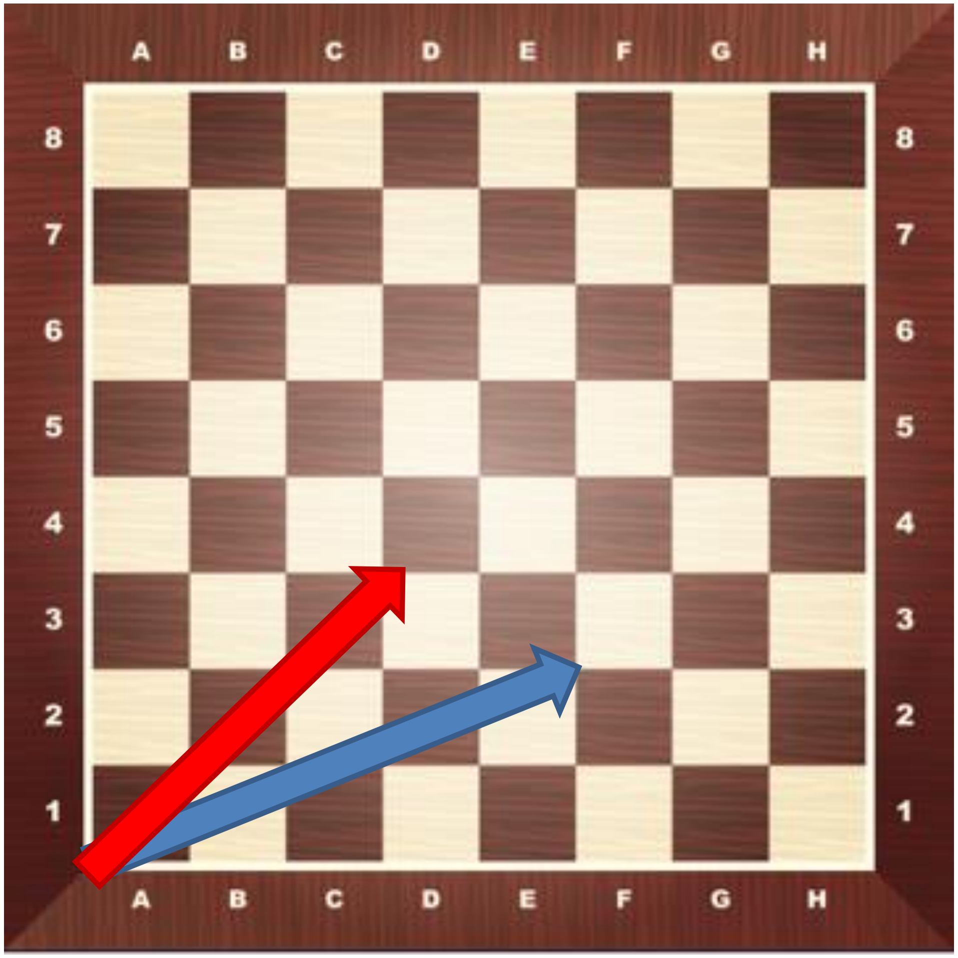


Basis of R^2

Standard Basis: {(1,0), (0,1)}

Other examples: {(5,0),(0,4)}, {(3,3),(5,2)}

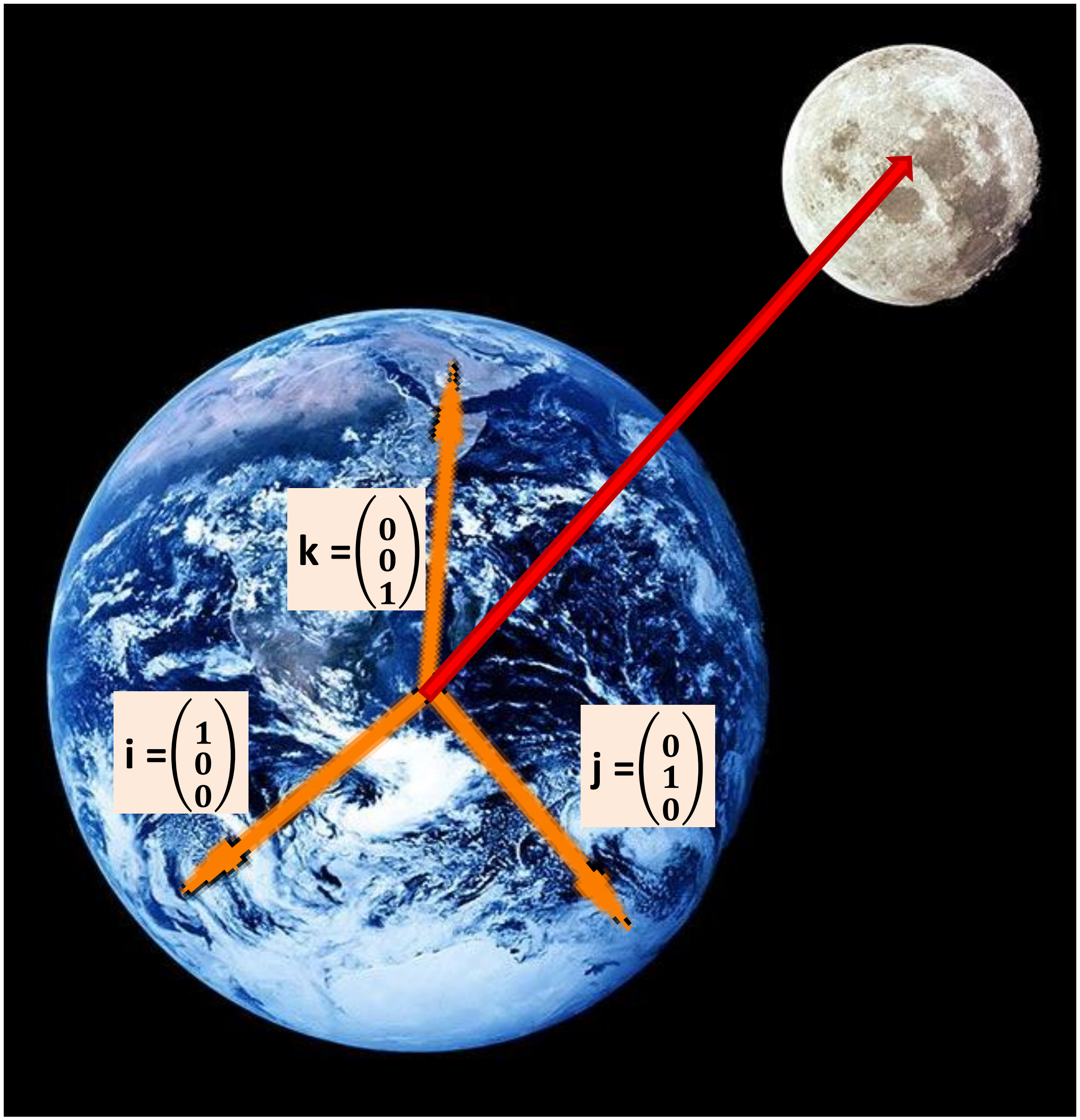




Each basis can provide a unique location of each boxes in a chess board.

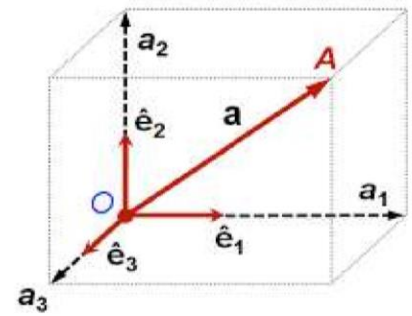
Basis of R^3

Standard Basis: {(1,0,0),(0,1,0),(0,0,1)}
Other examples: {(5,0,0),(0,4,0),(0,0,6)}, {(3,3,0),(5,0,2),(1,2,3)}



Considering center of earth as origin, we obtain locations of certain objects, satellites, like moon, asteroids, etc.

Basis of R^n



1. Basis of $\mathbb{R}^n$: $\boldsymbol{e_1} = \begin{pmatrix} 1 \\ 0 \\ . \\ . \\ 0 \end{pmatrix}$, $\boldsymbol{e_2} = \begin{pmatrix} 0 \\ 1 \\ . \\ . \\ 0 \end{pmatrix}$, $\ldots$, $\boldsymbol{e_n} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^n$ because (i) they are *linearly independent* (by inspection), and (ii) they *span* $\mathbb{R}^n$ because $c_1\boldsymbol{e_1} + c_2\boldsymbol{e_2} + \cdots c_n\boldsymbol{e_n} = \begin{pmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{pmatrix}$ generates any vector in $\mathbb{R}^n$ depending on the values of $c_i \forall i = 1,2,\ldots,n$

Few Common Examples of Bases

2. Let $\mathbb{P}_n$ be a vector space of all polynomial functions of degree n or less. The basis of $\mathbb{P}_n$ is $\{1, x, x^2, \dots, x^n\}$, the set of monomials.
(This is not a unique basis set because $\{p_0(x), p_1(x), \dots, p_n(x)\}$ also forms a basis where $p_i(x)$ is a polynomial in $\mathbb{P}_n$ of degree i.)

3. Let $\mathbb{M}_{m\times n}(\mathbb{F})$ denote the set of $m \times n$ matrices with entries in $\mathbb{F}$. Then $\mathbb{M}_{m\times n}(\mathbb{F})$ is a vector space over $\mathbb{F}$. Vector addition is just matrix addition and scalar multiplication is defined in the obvious way (by multiplying each entry of the matrix by the same scalar). The zero vector is just the zero matrix. One possible choice of basis is the matrices with a single entry equal to 1 and all other entries 0.
(We will study the vector space of matrices in more detail in subsequent lectures!)

Properties of Bases

1. Must every vector space have a basis?

*Ans: Every **non-zero, finitely generated** vector space has a basis!*

2. Does a vector space have a unique basis?

Ans: Usually a vector space will have many bases. e.g., the vector space $\mathbb{R}^2$ *has the basis* $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ *as well as the standard basis* $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$.

3. What is the dimension of a vector space?

Ans: $dim(\mathcal{V}) = no.\ of\ elements\ (vectors)\ in\ the\ basis\ (basis\ set)$.

Can you think of a vector space whose dimension is infinite?