Decision Making Under Uncertainty – An Introduction

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Decision Making in Engineering Design

PURDUE MECHANICAL ENGINEERING

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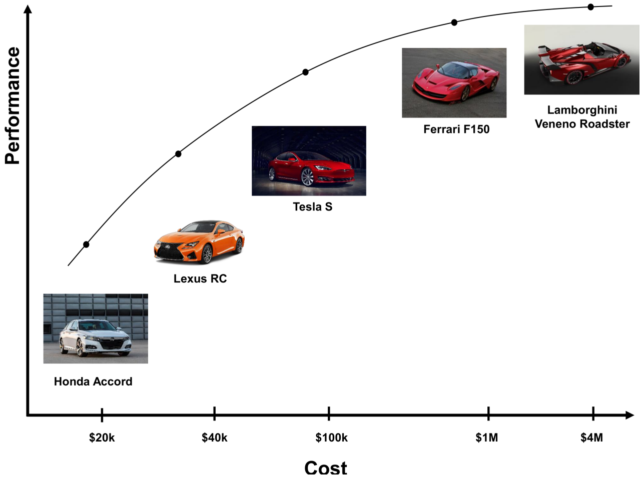
Learning Objectives

Learning Objectives for this week:

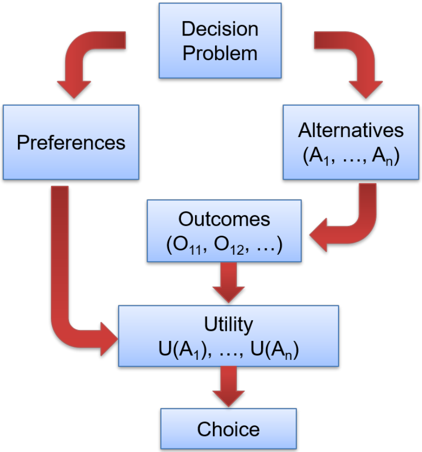
1. Gaining an awareness of the different types of decisions
2. Becoming familiar with the qualitative and quantitative characteristics of preferences
3. Learning how to use the mathematics of uncertainty to make decisions
4. Applying the decision making process to a decision problem (lab project)

Keeney, R. L. and H. Raiffa (1993). *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. Cambridge, UK, Cambridge University Press. Chapter 4.

Example
Decision to Buy a Car



The Structure of a Design Decision



Types of Decisions

	Single Attribute	Multiple Attributes
Certainty	I	II
Uncertainty	III	IV

Problem Statement for this Module

Choose among alternatives $\{A_1, A_2, \ldots\}$, each of which will eventually result in a consequence described by one attribute X which can take values $\{x_1, x_2, \ldots\}$.

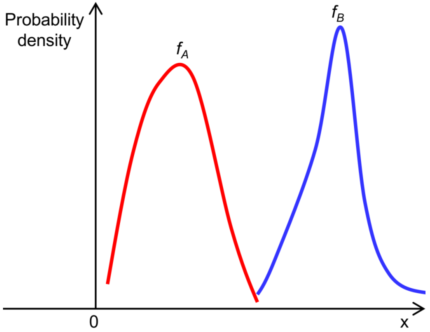
Decision maker **does not** know exactly what consequence (x_i) will result from each alternative.

But he/she **can** assign probabilities to the various consequences that might result from any alternative.

Alternate Approaches to the Risky Choice Problem

Alternate Approaches to Risky Choice Problem

(a) Probabilistic dominance



Is $A \prec B$ or $B \prec A$?

Alternate Approaches to Risky Choice Problem

(a) Probabilistic dominance (contd.)

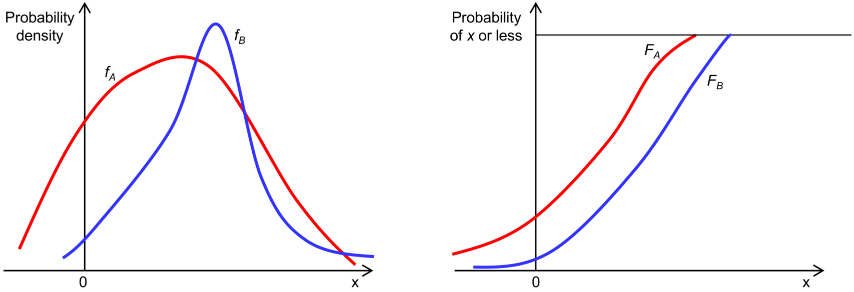
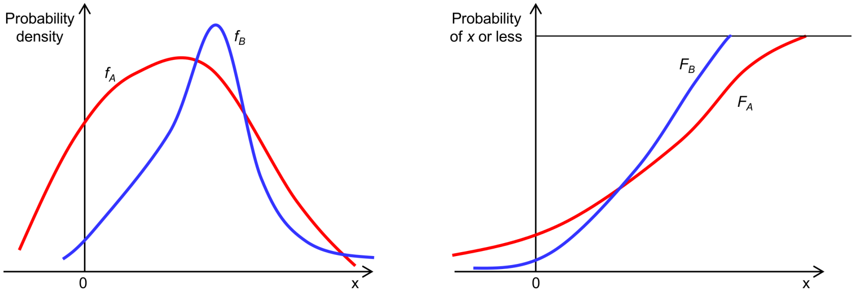


Figure: 4.2 on page 135 (Keeney and Raiffa)

Is $A \prec B$ or $B \prec A$?

Alternate Approaches to Risky Choice Problem

(a) Probabilistic dominance (contd.)



Is $A \prec B$ or $B \prec A$?

Alternate Approaches to Risky Choice Problem:
(b) Expected **"value"** of uncertain outcome

Consider the following alternatives

- A_1: Earn \$100,000 for sure
- A_2: Earn \$200,000 or \$0, each with probability 0.5
- A_3: Earn \$1,000,000 with probability 0.1 or \$0 with probability 0.9
- A_4: Earn \$200,000 with probability 0.9 or lose \$800,000 with probability 0.1

Are all alternatives equally desirable?

Note: The expected amount earned is exactly \$100,000 in each alternative.

Alternate Approaches to Risky Choice Problem

(c) Consideration of mean and variance

One possibility is to consider variance, in addition to the expected value of the outcome.

But, Alternatives A_3 and A_4 have the same mean and variance:

- A_3: Earn \$1,000,000 with probability 0.1 or \$0 with probability 0.9
- A_4: Earn \$200,000 with probability 0.9 or lose \$800,000 with probability 0.1

Are A_3 and A_4 equally preferred?

Limitations:

- Any measure that considers mean and variance only cannot distinguish between these two alternatives.
- Considering mean and variance imposes additional problem of finding relative preference between them.

Utility Theory

Primary Motivation for using Utility Theory

IF an appropriate utility is assigned to each possible consequence,
AND the expected utility of each alternative is calculated,

THEN the best course of action is the alternative with the highest *expected utility*.

Fundamentals of Utility Theory

Assume n consequences labeled $x_1, x_2, \ldots, x_n$ such that x_i is less preferred than x_{i+1}

$$x_1 \prec x_2 \prec x_3 \prec \cdots \prec x_n$$

Assume that for each i, the decision maker is indifferent between the following options:

1. Certainty Option

Receive x_i for sure

2. Risky Option

Receive x_n (best outcome) with probability π_i and x_1 (worst outcome) with probability $(1 - \pi_i)$. This option is denoted as $\langle x_n, \pi_i, x_1 \rangle$

Fundamentals of Utility Theory (contd.)

These π_i's can be thought of as numerical scaling of x's. The preference structure

$$x_1 \prec x_2 \prec x_3 \prec \cdots \prec x_n$$

maps to

$$\pi_1 < \pi_2 < \pi_3 < \cdots < \pi_n$$

Clearly,

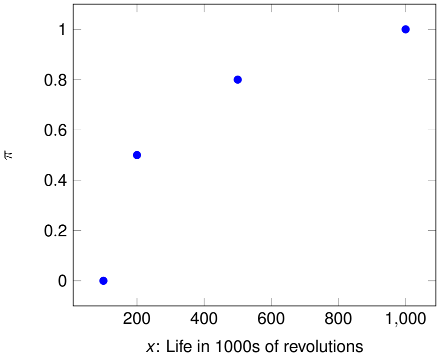
$$\pi_1 = 0$$
$$\pi_n = 1$$
$$\pi_1 < \pi_2 < \pi_3 < \cdots < \pi_n$$

Fundamentals of Utility Theory (contd.)

Example

Preferences:

- $x_1 = 100,000; \pi_1 = 0$
- $x_2 = 200,000; \pi_2 = 0.5$
- $x_3 = 500,000; \pi_3 = 0.8$
- $x_4 = 1,000,000; \pi_4 = 1$



Fundamentals of Utility Theory (contd.)

Fundamental Result of Utility Theory

The **expected value** of the π's can be used to numerically scale probability distributions over the x's.

The decision maker is to choose among probabilistic alternatives a' and a''

- a' : results in x_i with probability p'_i
  $\equiv \langle x_n, \bar{\pi}', x_1 \rangle$, i.e., $\bar{\pi}'$ chance at x_n and $(1 - \bar{\pi}')$ chance at x_1
- a'' : results in x_i with probability p''_i
  $\equiv \langle x_n, \bar{\pi}'', x_1 \rangle$ i.e., $\bar{\pi}''$ chance at x_n and $(1 - \bar{\pi}'')$ chance at x_1

The expected π values for alternatives a' and a'' are as follows:

$$\bar{\pi}' = \sum_i p'_i \pi_i \qquad \text{and} \qquad \bar{\pi}'' = \sum_i p''_i \pi_i$$

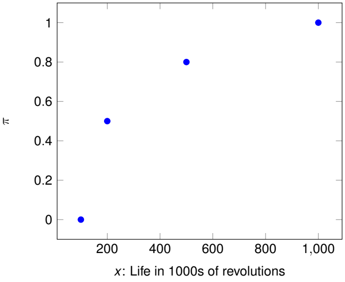
Now, we can rank order a', a'' in terms of $\bar{\pi}', \bar{\pi}''$

Expected Utility Calculation

Example with Discrete Outcomes

Preferences:

- $x_1 = 100,000; \pi_1 = 0$
- $x_2 = 200,000; \pi_2 = 0.5$
- $x_3 = 500,000; \pi_3 = 0.8$
- $x_4 = 1,000,000; \pi_4 = 1$



Decision: Choose among the following materials (a' and a''):

	p_1	p_2	p_3	p_4
Material a'	$p'_1 = 0.25$	$p'_2 = 0.25$	$p'_3 = 0.25$	$p'_4 = 0.25$
Material a''	$p''_1 = 0.2$	$p''_2 = 0.3$	$p''_3 = 0.3$	$p''_4 = 0.2$

Expected Utility Calculation

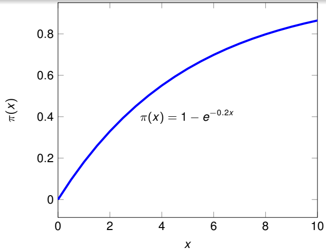
For continuous variables

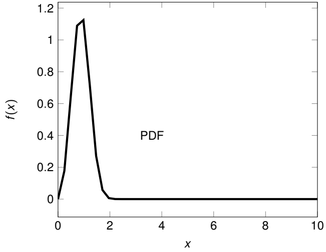
Expected utility of a random outcome **x**:

$$E[\pi(\mathbf{x})] = \int_{-\infty}^{\infty} \pi(x) f(x) dx$$

where, $f(x)$ is the probability density function, and $\pi(x)$ is the utility function.

Note: The utility function is typically denoted by $u(x)$.





Characteristics of Utility Functions

Qualitative Characteristics of Utility

The shape and functional form of the utility function tells us a great deal about the basic attitudes of the decision maker towards risk.

1. Monotonicity
2. Certainty equivalence
3. Strategic equivalence

Qualitative Characteristics of Utility

1. Monotonicity

Definition (Monotonicity)

For a monotonically increasing utility function

$$[x_1 > x_2] \Leftrightarrow [u(x_1) > u(x_2)]$$

For a monotonically decreasing utility function

$$[x_1 > x_2] \Leftrightarrow [u(x_1) < u(x_2)]$$

Can you think of an example where the utility is non-monotonic?

Qualitative Characteristics of Utility

2. Certainty Equivalence

Assume lottery L yields consequences $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$.

Define:
$\tilde{x}$: Uncertain consequence of lottery (i.e., random variable)
$\bar{x}$: Expected consequence

The **expected consequence** of the lottery is:

$$\bar{x} \equiv E(\tilde{x}) = \sum_{i=1}^{n} p_i x_i$$

The **expected utility** of the lottery is:

$$E[u(\tilde{x})] = \sum_{i=1}^{n} p_i u(x_i)$$

Qualitative Characteristics of Utility

2. Certainty Equivalence

Definition (Certainty equivalence)

A certainty equivalent of lottery L is the amount $\hat{x}$ such that the decision maker is indifferent between L and the amount $\hat{x}$ for certain.

$$u(\hat{x}) = E[u(\tilde{x})], \quad or \quad \hat{x} = u^{-1}Eu(\tilde{x})$$

Certainty equivalent of any lottery is unique for monotonic utility functions. For non-monotonic cases, the certainty equivalent may not be unique.

Qualitative Characteristics of Utility

2. Certainty Equivalence (continuous variables)

If x is a continuous variable, the associated uncertainty is described using a probability density function, $f(x)$. Then,

$$\bar{x} \equiv E(\tilde{x}) = \int x f(x) dx$$

The certainty equivalent $\hat{x}$ is a solution to

$$u(\hat{x}) = E[u(\tilde{x})] = \int u(x) f(x) dx$$

Qualitative Characteristics of Utility

2. Certainty Equivalence – Example

$u(x) = a - be^{-cx}$ Lottery $\langle x_1, 0.5, x_2 \rangle$

Determine:

- Expected consequence, $\bar{x}$
- Certainty equivalence, $\hat{x}$

Qualitative Characteristics of Utility

2. Certainty Equivalence – Example

$u(x) = a - be^{-cx}$

The lottery is described by the uniform probability density function: $f(x) = \dfrac{1}{x_2 - x_1}, \quad x_1 \leq x \leq x_2$

Determine:

- Expected consequence, $\bar{x}$
- Certainty equivalence, $\hat{x}$

Qualitative Characteristics of Utility

3. Strategic Equivalence

Definition (Strategic equivalence)

Two utility functions, u_1 and u_2, are strategically equivalent ($u_1 \sim u_2$) if and only if they imply the same preference ranking for any two lotteries.

If two utility functions are strategically equivalent, the certainty equivalents of two lotteries must be the same. Therefore,

$$u_1 \sim u_2 \Rightarrow u_1^{-1} E u_1(\tilde{x}) = u_2^{-1} E u_2(\tilde{x}), \quad \forall \tilde{x}$$

Qualitative Characteristics of Utility

3. Strategic Equivalence (contd.)

For some constants h and $k > 0$, if

$$u_1(x) = h + ku_2(x), \quad \forall x$$

then $u_1 \sim u_2$

Theorem

If $u_1 \sim u_2$, there exists two constants h and $k > 0$ such that

$$u_1(x) = h + ku_2(x), \quad \forall x$$

Example: $u(x) = a + bx \sim x, b > 0$

We can show that if the utility function is linear, the certainty equivalent for any lottery is equal to the expected consequence of that lottery.

Risk Aversion – An Illustration

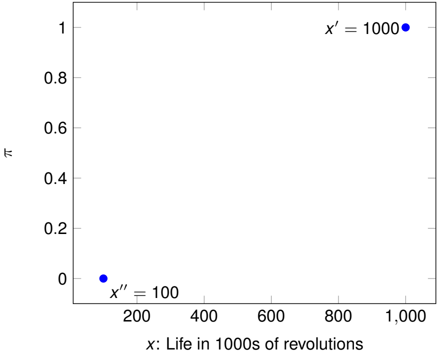
Consider a lottery $\tilde{x} = \langle x', 0.5, x'' \rangle$.

The expected consequence of the lottery is:

$$\bar{x} = E(\tilde{x}) = 0.5x' + 0.5x'' = \frac{x' + x''}{2}.$$

Choose between:

- $\bar{x}$ for certain, and
- lottery $\tilde{x} = \langle x', 0.5, x'' \rangle$



If the decision maker prefers the certain outcome $\bar{x}$, then the decision maker prefers to *avoid risks* $\Rightarrow$ Risk Averse.

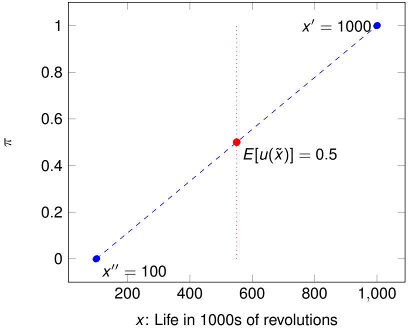
Definition of Risk Aversion

Definition (Risk Aversion)

A decision maker is risk averse if he prefers the expected consequence of any non-degenerate lottery to that lottery.

Let the possible consequences of any lottery are represented by $\tilde{x}$, a decision maker is risk averse if, for all non-degenerate lotteries, utility of expected consequence is greater than the expected utility of that lottery, i.e.,

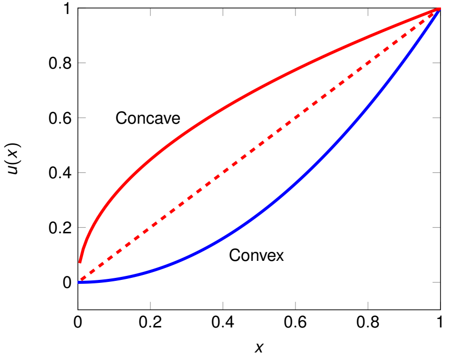
$$u[E(\tilde{x})] > E[u(\tilde{x})]$$



Risk Aversion and Utility Functions

Theorem

A decision maker is risk averse if and only if his/her utility function is concave.



A Measure of Risk Aversion

Definition (Risk aversion)

The local risk aversion at x, written $r(x)$, is defined by

$$r(x) = -\frac{u''(x)}{u'(x)}$$

- $r(x) > 0 \Rightarrow$ Risk Averse
- $r(x) < 0 \Rightarrow$ Risk Prone

Characteristics of this measure:

1. it indicates whether the utility function is risk averse or risk prone
2. shows equivalence between two strategically equivalent utility functions

A Measure of Risk Aversion

Example

Determine $r(x)$ for:

$$u(x) = a - be^{-cx}$$

Risk Prone and Risk Neutral

Definition (Risk Prone)

A decision maker is risk prone if (s)he prefers any non-degenerate lottery to the expected consequence of that lottery.

$$u[E(\tilde{x})] < E[u(\tilde{x})]$$

Definition (Risk Neutral)

A decision maker is risk neutral if (s)he is indifferent between any non-degenerate lottery and the expected consequence of that lottery.

$$u[E(\tilde{x})] = E[u(\tilde{x})]$$

Risk Premium

Definition (Risk Premium of a lottery)

The **risk premium** (*RP*) of a lottery $\tilde{x}$ is its expected value ($\bar{x}$) minus its certainty equivalent ($\hat{x}$).

$$RP(\tilde{x}) = \bar{x} - \hat{x} = E(\tilde{x}) - u^{-1}Eu(\tilde{x})$$

The **risk premium** (*RP*) is the amount of the attribute that a (risk averse) decision maker is willing to "give up" from the average to avoid the risks associated with the particular lottery.

Risk Premium

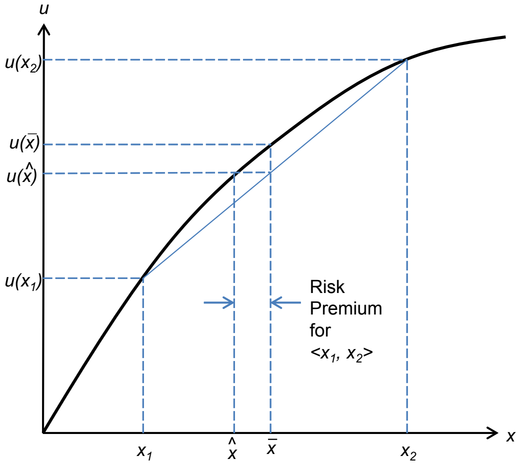


Figure: 4.5 on page 152 (Keeney and Raiffa)

Insurance Premium

The **insurance premium** (*IP*) for a lottery $\tilde{x}$ is the negative of the certainty equivalent of the lottery.

$$IP(\tilde{x}) = -\hat{x} = -u^{-1}Eu(\tilde{x})$$

The insurance premium is the amount that the decision maker is willing to give up to rid himself of the financial responsibility of the lottery.

Risk Premium

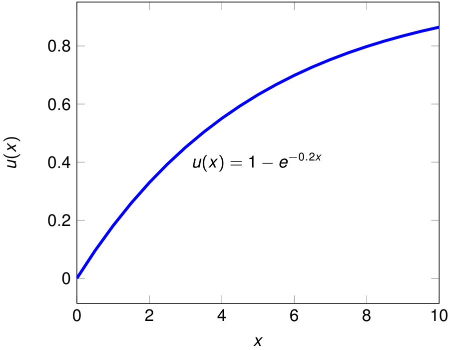
Example

For the utility function

$$u(x) = 1 - e^{-0.2x}$$

Determine the following for the lottery $\tilde{x} = \langle 10, 0.5, 0 \rangle$

- Expected consequence ($\bar{x}$)
- Certainty equivalent ($\hat{x}$)
- Risk premium (RP)
- Insurance premium (IP)



Monotonically Decreasing Utility Functions

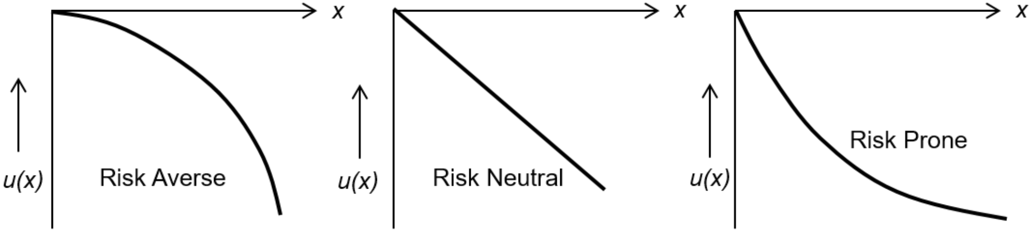


Figure: 4.15 on page 180 (Keeney and Raiffa)

A measure of risk aversion for a decreasing utility function is

$$q(x) \equiv \frac{u''(x)}{u'(x)} = \frac{d}{dx}[\log(u'(x))]$$

Note: This is same as $r(x)$, except the $-$ve sign. Positive value of $q(x)$ means that the decision maker is risk averse.

Non-monotonic Utility Functions

For non monotonic preferences, a decision maker is risk averse [risk prone] if and only if his utility function is concave [convex].

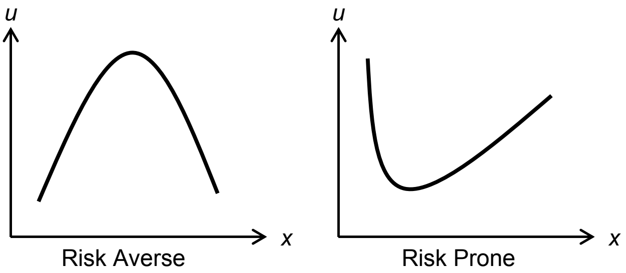


Figure: 4.18 on page 188 (Keeney and Raiffa)

For non-monotonic utility functions, the certainty equivalent is not necessarily unique. The risk premium and measure of risk aversion cannot be usefully defined.

Summary

1. Alternate Approaches to the Risky Choice Problem
2. Utility Theory
3. Characteristics of Utility Functions
   - Qualitative Characteristics of Utility

Keeney, R. L. and H. Raiffa (1993). *Decisions with Multiple Objectives: Preferences and Value Tradeoffs*. Cambridge, UK, Cambridge University Press. Chapter 4.