Mathematics of Uncertainty

Lead instructor: Amrik Sen

Co-instructors: Sanjay Bose
Jitesh Panchal (Module 5.2)

Instructional Manager: Vanita Sharma

Teaching Assistants: Arpit Shukla
Manraj Singh Ghuman
Sandeep Kumar

Tutorials
Worksheet
Mini-projects

Mathematics of Uncertainty

Module 1: Probability Basics and Definitions	(6)
Module 2: Probability Distributions	(6)
Module 3: Discrete Time Markov Chains	(6)
Module 4: Statistical Experiments	(6)
Module 5: Statistics for Complex Problems	(4)

Each module consists of Lectures and a Mini-Project

Study of Probability in Ancient India

Movie Recommendation for the Weekend

Watch

"Breaking Vegas: The True Story of the MIT Blackjack Team"
(*Documentary*)

or

"21"
(*Hollywood Movie*)

Both Available on YouTube

Shows what you can do with Mathematics and Probabilities

Birthday Surprise

In a party, make a bet that there will be at least two people in the room with the same birthday!

If the number of people in the party is more than 23, you are likely to win the bet!

Textbook: “*Practical Introduction to Probability and Statistics – A project based conceptual guide to students and practioners*”, by Amrik Sen

Draft Edition, under peer review by Cambridge University Press, will be provided to students on course website

Reference Books:

1. “Weighing the Odds - A course in Probability and Statistics”, by David Williams.
2. “Probability Theory – The Logic of Science”, by E.T. Jaynes.
3. “Introduction to Probability and Statistics for Engineers and Scientists”, by Sheldon M. Ross.
4. “Probability, Random Variables and Stochastic Processes”, Papoulis and Pillai

Assessments:

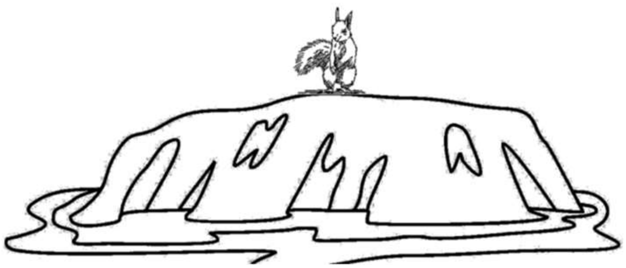
- Two written exams: 30% Mid Term (proctored) + 20% End Term (proctored) TOTAL Weightage = 50%
- Five Mini-Projects in the laboratory of 10% each TOTAL Weightage 50%

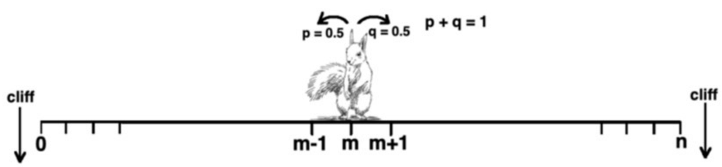
Note that students will have to obtain a “pass” grade separately for the written exam (theory) and the laboratory exams (projects) in order to pass the course.

For details, consult the “Course Brochure” provided in the course website.

The laboratory experiments and projects will use MATLAB (all modules) and also Python (for module 5.2)

Misadventures of Squeaky





Our friend Squeaky is trapped somewhere in the middle of a lonely island hill with sharp cliffs on both sides. Squeaky is excited and jumps around in her merry way. At any given instance, she decides to hop to the left or to the right independent of her past moves. Squeaky is unaware of the impending danger of falling off the cliff.

In this project, we will use calculations based on the principles of conditional probability, the law of total probability, and the law of total expectation to predict her fate. In other words, what are the odds that she will bounce around on the island hill, her left-sided moves balancing out her right-sided moves on an average, and never actually trip and fall off on either side? Or will chance play the devil's role and will she eventually drift off to one side and perish? And if the latter turns out to be true, then what is her life expectancy in terms of the total number of hops starting from her first move? Does a certain initial position on the hill give her the best chance to survive the longest?

In addition to our theoretical calculations, we will also build a computer simulation of her actions to corroborate our result. For convenience, we shall assume that the island is one dimensional, i.e., Squeaky's movements are restricted exclusively to lateral directions(left or right). While we build the computer-simulated solution, we will learn to apply a random number generator using a computer software in order to mimic Squeaky's mental choices to hop either to the left or to the right independent of her past moves.

Deterministic Outcomes

Permutations	*when the order matters*
Combinations	*when the order does not matter*

Permutations:

* Number of permutations of n different things $n! = n \times (n-1) \times \ldots \times 2 \times 1$

* Consider a situation where there are a total of n objects of r different types and where the number of objects of type k is n_k with $k = 1, 2, \ldots\ldots, r$

and $n = n_1 + n_2 + \ldots\ldots + n_r$

If we assume that the objects of the same type are indistinguishable from each other then the number of different ways in which the objects can be arranged is

$$\frac{n!}{n_1!\,n_2!\ldots\ldots n_r!}$$

* If we have n different things then the number of permutations we get by taking only r ($r \leq n$) of them at a time are $P_r^n = n(n-1)\ldots\ldots(n-r+1) = \frac{n!}{(n-r)!}$

For example, if we have nine slots and four differently colored balls, then the number of different arrangements possible are $\frac{9!}{5!} = 3024$

Combinations:

Consider when the order of arrangement is not important.

For example when we want to choose *r* items out of *n* in any order, then the number of ways in which this can be done is given by

$$C_r^n = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

For example, if we have nine slots and four identically colored balls, then the number of different arrangements possible are $\binom{9}{4}$=126

Useful Result: $C_r^n(r!) = P_r^n$

Permutations and Combinations, as discussed in the previous slides, belong to a class of experiments that have deterministic outcomes, i.e. there are a finite and fixed of ways of arranging or combining items.

There may be experiments where the outcomes are not certain. For example, if we consider the experiment of putting five differently colored balls in five slots, we may ask what is the probability that the first slot is filled with a ball of a specific color.

This is where the study of probability and statistics comes in.

Probability: Giving a measure of likelihood that an event will occur

Statistics: Dealing with the collection (sampling), organization, analysis and interpretation of data to make inferences and forecasts. (It relies on the principles of probability.)

Some Definitions

Probability: The Probability of an Event is the measure of the likelihood that the event will occur, e.g. *the probability that it will rain today is 0.75*

Statistics: This is the branch of mathematics that deals with the collection (sampling), organization, analysis and interpretation of data including making inferences and forecasts.

Probability deals with predicting the likelihood of future events, while statistics involves the analysis of the frequency of past events

Probability Space (Definition):

A probability space is a triple $(\Omega, \mathcal{F}, P)$ where Ω is a sample space, $\mathcal{F}$ is a $\sigma -$ algebra (sigma algebra) of events and *P* is a probability measure on $\mathcal{F}$

- The sample space Ω is the set of all possible outcomes of a probabilistic experiment
- The $\sigma -$ algebra $\mathcal{F}$ is the collection of all subsets of Ω to which we are able/willing to assign probabilities; these subsets are called events
- The probability measure *P* is a function that associates a probability to each of the events belonging to the $\sigma -$ algebra $\mathcal{F}$

Example: Probabilistic experiment to get a ball from an urn containing two balls, one red (*R*) and one blue (*B*)

$\Longrightarrow$ Sample Space $\Omega = \{R, B\}$ and a possible $\sigma -$ algebra $\mathcal{F}$ of events is $\mathcal{F} = \{\emptyset, \Omega, \{R\}, \{B\}\}$ where

$\emptyset$ is the empty set (nothing happens)	$P(F) = 0$ if $F = \emptyset$
Ω**:** either *R* or *B* is extracted	= 0.5 if *F*={*R*}
{*R*}: Red ball is extracted	= 0.5 if *F*={*B*}
{*B*}: Blue ball is extracted	= 1 if *F*= Ω

Axioms of Probability

Axioms are regarded as *a priori propositions* whose veracity is accepted universally without requiring validation by demonstration, i.e. they are accepted without proof.

Axioms are useful as they allow deduction of realizable experiences

1. $P(E) \geq 0$ for all $E \in \mathcal{F}$ (non-negativity of probability)

2. $P(\Omega)$ = 1 (unitarity)

3. $P(\bigcup_{i=1}^{\infty} E_i)$=$\sum_{i=1}^{\infty} P(E_i)$ for a countable sequence of disjoint events E_1, E_2 ……
(*σ*-additivity)

Supplementary Properties of the probability measure *P* that are helpful while performing calculations

1. For $E_1, E_2 \in \mathcal{F}$, we have $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$
   This can be generalized to *n* events E_1, E_2 …….E_n by induction. Principle of Inclusion-Exclusion

2. If E_1, E_2 are independent events (i.e. $E_1 \perp E_2$), then $P(E_1 \cap E_2) = P(E_1)P(E_2)$

3. If A^c is the event complementary to the event *A*, then $P(A^c) = 1 - P(A)$

4. The probability of the *impossible event* is zero, i.e $P\{\emptyset\} = 0$

5. It is important to distinguish between *mutually exclusive* (disjoint) events and *independent* events

Mutually Exclusive	if $E_1 \cap E_2 = \{\emptyset\} \Rightarrow P(E_1 \cap E_2) = 0$
Independent Events	$P(E_1 \cap E_2) = P(E_1)P(E_2)$ and $P(E_1 \mid E_2) = P(E_1)$

Mutually Exclusive events cannot happen concurrently.
Independent Events may happen concurrently but the outcome of one does not affect the outcome of the other