# Lecture Notes: Review of Matrices and Their Operations

## 1. Matrix Definiton

1.1. **Definition (Matrix).** A matrix is an ordered rectangular array of numbers( or functions). The numbers (or functions) are called the elements or the entries of the matrix.

**Example** $A = \begin{bmatrix} -2 & 5 \\ 0 & \sqrt{5} \\ 3 & \sqrt{6} \end{bmatrix}, B = \begin{bmatrix} 2+i & 3 & \frac{-1}{2} \\ 3.5 & -1 & 2 \\ \sqrt{3} & 5 & \frac{5}{4} \end{bmatrix}, C = \begin{bmatrix} 1+x & x^3 & 3e^x \\ \cos(x) & \sin(x)+2 & \tan(x) \end{bmatrix}$

1.2. **Order/Size of a matrix.** A matrix having $m$ rows and $n$ columns is called a matrix of order $m \times n$.

**Example** $\begin{bmatrix} 3 & 7 & 2 \end{bmatrix}$ is a matrix of order $1 \times 3$ and $\begin{bmatrix} 9 & 13 & 5 \\ 1 & 11 & 7 \\ 2 & 6 & 3 \end{bmatrix}$ is a matrix of order $3 \times 3$.

1.3. **Notation of a matrix.** A general $m \times n$ matrix has the following rectangular array

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}$$

$$OR$$

$$A = [a_{ij}]_{m \times n} \quad 1 \le i \le m, 1 \le j \le n, \quad i, j \in \mathbb{N}$$

## 2. Types of Matrices

2.1. **Row Matrix.** A matrix with only one row. **Example:**

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

2.2. **Column Matrix.** A matrix with only one column. **Example:**

$$B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

2.3. **Square Matrix.** A matrix with the same number of rows and columns. **Example:**

$$C = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

2.4. **Diagonal Matrix.** A square matrix where non-diagonal elements are zero. **Example:**

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

2.5. **Identity Matrix.** A square matrix where diagonal elements are 1, and non-diagonal elements are 0. **Example:**

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2.6. **Zero Matrix.** A matrix where all elements are zero. **Example:**

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

## 3. Addition of Matrices

For two matrices $A$ and $B$ of the same order $m \times n$, their sum $C$ is given by:

$$C = A + B,$$

where each element is computed as $c_{ij} = a_{ij} + b_{ij}$.

**Example:**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

### 3.1. Addition of matrices.

Two matrices $A$ and $B$ can be added if they are of the same order. The matrices $A$ and $B$ are denoted as,

$$A = [a_{ij}]_{m\times n} \quad B = [b_{ij}]_{m\times n} \quad 1 \leq i \leq m, 1 \leq j \leq n, \quad i, j \in \mathbb{N}$$

then $\exists\ C = A + B$ of order $m \times n \ni C = [a_{ij} + b_{ij}]_{m\times n} = [c_{ij}]_{m\times n}$

**Example** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Here, $A + B$ doesn't exists since A and B are of different orders.

Although, $A + C = \begin{bmatrix} 1+1 & 2+0 \\ 3+0 & 4+1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}$

## 4. Product of two matrices

The product $AB$ of matrices $A$ and $B$ is defined if and only if the number of columns of $A$ is equal to the number of rows of $B$ ;

Consider, $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$,
then,

$$AB = C = [c_{ij}]_{m\times p}$$

where, $C$ is of order $m \times p$ and

$$c_{ij} = \sum_{j=1}^{n} a_{ij} b_{jk}$$

### 4.1. Mathematical Representation of Matrix Multiplication:

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mj} & \dots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1j} & \dots & b_{1p} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2j} & \dots & b_{2p} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ b_{i1} & b_{i2} & b_{i3} & \dots & b_{ij} & \dots & b_{ip} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nj} & \dots & b_{np} \end{bmatrix}_{n\times p}$$

where, (say) the element $c_{22}$ (of $AB$) $= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} + \ldots + a_{2n}b_{n2}$

**Example** Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \quad B = \begin{bmatrix} 1 & 2 \end{bmatrix}_{1\times 2}$

Here, $AB$ does not exists.

whereas when, $A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 5 \end{bmatrix}_{3\times 2} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{2\times 3}$

then,

$$AB = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 3 \\ 4 & 5 & 4 \end{bmatrix}_{3\times 3}$$

4.2. **Vector Multiplication.** For a row vector

$$v = [a \quad b \quad c]$$

and a column vector

$$w = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

their dot product is given by:

$$v \cdot w = ax + by + cz.$$

**Example** Given the vectors:

$$v = [1 \quad 2 \quad 3], \quad w = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

their dot product is computed as:

$$v \cdot w = (1 \times 4) + (2 \times 5) + (3 \times 6) = 4 + 10 + 18 = 32.$$

## 5. Transpose of a matrix

Consider $A = [a_{ij}]_{m\times n}$ then the transpose of an $m \times n$ matrix is defined to be a matrix of order $n \times m$ denoted by $A^{\mathrm{T}}$ such that

$$A^{\mathrm{T}} = [a_{ij}]_{n\times m} \quad \forall \quad i, j$$

**Example** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ then, $\quad A^{\mathrm{T}} = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$

ie. The rows of matrix $A$ becomes the columns of new matrix $A^{\mathrm{T}}$ or vice - versa.

5.1. **Properties of Transpose.** The following are some important properties of the transpose of a matrix:

(1) **Transpose of a Transpose:**

$$(A^T)^T = A$$

(2) **Transpose of a Sum:**

$$(A + B)^T = A^T + B^T$$

(3) **Transpose of a Product:**

$$(AB)^T = B^T A^T$$

## 6. Symmetric and Skew-Symmetric Matrices

6.1. **Symmetric Matrices.** A square matrix $A$ is said to be **symmetric** if it is equal to its transpose, i.e.,

$$A^T = A.$$

This means that the entries of $A$ satisfy $a_{ij} = a_{ji}$ for all $i, j$.

**Example:**

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 4 & 5 \\ 1 & 5 & 6 \end{bmatrix}$$

Since $A^T = A$, this matrix is symmetric.

6.2. **Skew-Symmetric Matrices.** A square matrix $A$ is said to be **skew-symmetric** if its transpose is equal to its negative, i.e.,

$$A^T = -A.$$

This means that the entries satisfy $a_{ij} = -a_{ji}$ for all $i, j$, and all diagonal elements must be zero ($a_{ii} = 0$).

**Example:**

$$B = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$$

Since $B^T = -B$, this matrix is skew-symmetric.

## 7. Determinants

Determinants are defined as a scalar quantity associated with a square matrix denoted by $\det A$ or $|A|$. Determinant of a $2 \times 2$ matrix is defined as

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Similarly, Determinant of a $3 \times 3$ matrix is defined as

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11} \left(a_{22}a_{33} - a_{23}a_{32}\right) - a_{12} \left(a_{21}a_{33} - a_{23}a_{31}\right) + a_{13} \left(a_{21}a_{32} - a_{22}a_{31}\right).$$

Each determinant of a $2 \times 2$ matrix in this equation is called a minor of the matrix $A$.

**Example** (i)$A = \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix} \Longrightarrow \det(A) = -10$

(ii)$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ **then** $\det(A) = 1 \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1(6-2) - 2(4-1) + 3(4-3) = 4 - 6 + 3 = 1$

### 7.1. Geometrical Interpretation of Determinants.

- **For a** $2 \times 2$ **matrix** $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: The determinant $\det(A) = ad - bc$ represents the *signed area* of the parallelogram formed by the column vectors of $A$ in 2D space.
- **For a** $3 \times 3$ **matrix** $A$: The determinant represents the *signed volume* of the parallelepiped formed by the three column vectors in 3D space.

**Meaning of Determinant Zero**

- **For a** $2 \times 2$ **matrix**, $\det(A) = 0$ means that the two column vectors are *linearly dependent*, meaning they lie on the same line, so the parallelogram collapses to a line (zero area).
- **For a** $3 \times 3$ **matrix**, $\det(A) = 0$ means the three column vectors are *coplanar*, meaning they lie in the same plane, so the parallelepiped collapses to a flat shape (zero volume).

In both cases, the matrix is *singular*, meaning it does not have an inverse and does not define a full-rank transformation.

### 7.2. Properties of Determinants.

7.2.1. *Determinant of Identity Matrix.*

$$\det(I_n) = 1$$

The determinant of the identity matrix is always 1.

7.2.2. *Determinant of a Triangular Matrix.* If $A$ is a triangular (upper or lower) or diagonal matrix, then its determinant is the product of its diagonal elements:

$$\det(A) = a_{11}a_{22} \cdots a_{nn}$$

7.2.3. *Swapping Rows or Columns Changes Sign.* If two rows (or columns) of a matrix are interchanged, the determinant changes sign:

$$\det(B) = -\det(A)$$

where $B$ is obtained by swapping two rows or columns of $A$.

7.2.4. *Multiplying a Row or Column by a Scalar.* If a row (or column) of $A$ is multiplied by a scalar $c$, then:

$$\det(B) = c\det(A)$$

where $B$ is the new matrix.

7.2.5. *Determinant of a Product.* The determinant of the product of two matrices equals the product of their determinants:

$$\det(AB) = \det(A) \cdot \det(B)$$

7.2.6. *Determinant of a Transpose.* The determinant of a matrix is equal to the determinant of its transpose:

$$\det(A^T) = \det(A)$$

7.2.7. *Determinant of an Invertible Matrix.* If $A$ is invertible, then its determinant is nonzero, and:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

7.2.8. *Determinant of a Singular Matrix.* If $\det(A) = 0$, then $A$ is singular (non-invertible).

7.2.9. *Addition of Rows or Columns Does Not Change Determinant.* Adding a multiple of one row (or column) to another does not change the determinant.

7.2.10. *Determinant of Block Matrices.* If $A$ and $B$ are square matrices and the block matrix is of the form:

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

then:

$$\det\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \det(A) \cdot \det(B)$$

## 8. Minor & Cofactors

8.1. **Minor.** Let $A$ be a matrix of order $n \times n$ then the minor of element $a_{ij}$ is equal to det of a submatrix of order $(n-1) \times (n-1)$ which is obtained by leaving $i^{th}$ row and $j^{th}$ column of $A$. It is denoted by $M_{ij}$.

**Example** For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$(leaving $1^{st}$ row and $1^{st}$ column of A)

8.2. **Co-factor matrix.** Let $A = (a_{ij})_{n\times n}$ be a matrix then the co-factor of element $a_{ij} = A_{ij} = (-1)^{i+j} M_{ij}$.

8.3. **Adjoint of a matrix.** Let $A = [a_{ij}]_{n\times n}$ and its Co-factor matrix is $C = [A_{ij}]_{n\times n} = \left[(-1)^{i+j} M_{ij}\right]$

The transpose of the Co-factor matrix of A is known as the adjoint of $A$, denoted by $adj\,(A) = [A_{ij}]^{\mathrm{T}} = C^{\mathrm{T}}$

i.e if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nj} & \dots & a_{nn} \end{bmatrix}_{n\times n}$$

then

$$adj\,(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} & \dots & A_{j1} & \dots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \dots & A_{j2} & \dots & A_{n2} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ A_{1i} & A_{2i} & A_{3i} & \dots & A_{ji} & \dots & A_{ni} \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ A_{1n} & A_{2n} & A_{3n} & \dots & A_{jn} & \dots & A_{nn} \end{bmatrix}_{n\times n}$$

where $A_{ij}$ is the Co-factor element of $a_{ij}$

**Example** Find the $adj(A)$ for $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$

**Solution:-** First we find the co-factor matrix.

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 6 \\ 0 & 3 \end{vmatrix} = 6, \qquad A_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 6 \\ 0 & 3 \end{vmatrix} = 0, \qquad A_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 5 \\ 0 & 3 \end{vmatrix} = -12, \qquad A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ 0 & 3 \end{vmatrix} = 3, \qquad A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 4 \\ 0 & 0 \end{vmatrix} = 0,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 5 \\ 2 & 6 \end{vmatrix} = 24 - 10 = 14, \qquad A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 0 & 6 \end{vmatrix} = -6, \qquad A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2.$$

hence the Co-factor matrix of A,

$$C = \begin{bmatrix} 6 & 0 & 0 \\ -12 & 3 & 0 \\ 14 & -6 & 2 \end{bmatrix}$$

then

$$adj(A) = C^T = \begin{bmatrix} 6 & -12 & 14 \\ 0 & 3 & -6 \\ 0 & 0 & 2 \end{bmatrix}$$

## 9. Inverse of a square matrix

$A_{n\times n}$ is said to be invertible if $\exists\ B_{n\times n}$ such that $AB = BA = I_n$ where $I_n$ is the identity matrix of order $n \times n$. If $B$ exists, it is denoted by $A^{-1}$ and given by

$$A^{-1} = \frac{1}{\det(A)}(adj A) \qquad (A^{-1} \text{ exist } \iff \det(A) \neq 0).$$

**Example**. Find the inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc \neq 0$.

**Solution:-**Since given that $\det(A) = ad - bc \neq 0$ so $A$ is a invertible matrix.

$$A_{11} = (-1)^{1+1}d = d, \qquad A_{12} = -c, \qquad A_{21} = -b, \qquad A_{22} = a.$$

so the Co- factor matrix is

$$[A_{ij}] = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

and

$$adj A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hence,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

**Example** If exist, find the inverse of the matrix $A = \begin{bmatrix} 5 & 0 & 7 \\ 2 & 1 & 3 \\ 0 & 4 & 6 \end{bmatrix}$.

**Solution:-** $\det(A) = 5(6 - 12) + 7(8) = -30 + 56 = 26 \neq 0$

Hence $A$ is invertible.

$$\text{Co-factor matrix } ([A_{ij}]) = \begin{bmatrix} -6 & -12 & 8 \\ 28 & 30 & -20 \\ -7 & -1 & 5 \end{bmatrix}$$

$$adj(A) = \text{Transpose of the Co-factor matrix } = \begin{bmatrix} -6 & 28 & -7 \\ -12 & 30 & -1 \\ 8 & -20 & 5 \end{bmatrix}$$

Hence,

$$A^{-1} = \frac{1}{\det(A)}(adj(A)) = \frac{1}{26} \begin{bmatrix} -6 & 28 & -7 \\ -12 & 30 & -1 \\ 8 & -20 & 5 \end{bmatrix}$$

**Example** If exist, Find the inverse of $\begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix}$

$\det(A) = 16 - 16 = 0$ . Hence $A$ is not invertible.

### 9.1. **Properties of Inverse of Matrix.**

9.1.1. *Existence of Inverse.* A square matrix $A$ has an inverse if and only if $\det(A) \neq 0$.

9.1.2. *Uniqueness of Inverse.* If a matrix $A$ is invertible, then its inverse $A^{-1}$ is unique.

9.1.3. *Inverse of a Product of Matrices.* If $A$ and $B$ are both invertible matrices, then:

$$(AB)^{-1} = B^{-1}A^{-1}$$

9.1.4. *Inverse of a Transpose.* The inverse of the transpose of a matrix is the transpose of the inverse:

$$(A^T)^{-1} = (A^{-1})^T$$

9.1.5. *Inverse of a Scalar Multiple.* If $A$ is an invertible matrix and $c$ is a scalar, then:

$$(cA)^{-1} = \frac{1}{c}A^{-1}$$

9.1.6. *Inverse of the Inverse.* The inverse of the inverse of a matrix is the matrix itself:

$$(A^{-1})^{-1} = A$$

9.1.7. *Inverse of a Diagonal Matrix.* If $D$ is a diagonal matrix with non-zero diagonal elements, then its inverse is also a diagonal matrix with the reciprocal of each diagonal element:

$$D^{-1} = \text{diag}\left(\frac{1}{d_1}, \frac{1}{d_2}, \ldots, \frac{1}{d_n}\right)$$

where $d_i$ are the diagonal elements of $D$.