Linear Ordinary Differential Equation with Constant Coefficients

Amrik Sen

Engineering Mathematics in Action: FM112

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Lecture Plan

Topic: Solving linear differential eqs. with constant coefficients

1. Conceptual introduction
2. Solution technique via examples
3. Ground work for harder problems

Form of Linear ODE w/ const. coeff.: $\mathscr{L}y(x) = 0$

$$\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1\frac{dy}{dx} + a_0 y = 0 \quad (1)$$

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0 \quad (2)$$

$\mathscr{L} := a_0 + a_1\frac{d}{dx} + ... + a_{n-1}\frac{d^{n-1}}{dx^{n-1}} + \frac{d^n}{dx^n}$ is the *linear differential operator*.

Solution form

Seek a solution of the form $\boxed{y(x) = e^{rx}}$ (**Why?**)

$\mathscr{L}(e^{rx}) = \mathscr{P}(r)e^{rx}$, where $\mathscr{P}(r) = r^n + a_{n-1}r^{n-1} + ... + a_1 r + a_0$.

Characteristic equation

$$\boxed{\mathscr{P}(r) = r^n + a_{n-1}r^{n-1} + ... + a_1 r + a_0 = 0}$$

Solution set depends on the nature of roots of characteristic eq.

1. n distinct real roots: $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} + ... + c_n e^{r_n x}$
2. all real but some multiple roots: eg. $m <= n$ multiple roots $r = r_0$, remaining roots are distinct; then
   $y(x) = (c_1 + c_2 x + ... + c_m x^{m-1}) e^{r_0 x} + d_1 e^{r_1 x} + ... + d_{n-m} e^{r_{(n-m)} x}$
   Thought exercise: justify why above is true?
   hint: $y = u(x) e^{r_0 x}$.
3. complex roots:
   $y = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) +$ linear combination of real solutions

   For repeated complex roots, follow the prescription in (2).

Example: ODEs with constant coeff. (distinct real roots)

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Thus $\boxed{y(t) = 3e^{-2t} - 2e^{-3t}}$.

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The solution will be of the form:
$y(t) = (c_1 + c_2 t + c_3 t^2)e^{-t} + (c_4 + c_5 t)$ ...

HW: complex roots

Question: Solve the ODE $\frac{d^4y}{dt^4} + 8\frac{d^2y}{dt^2} + 16y = 0$ with some appropriate initial conditions.

You may leave your answer in terms of constants of the problem.

Example: solving ODE with const. coeff.

Problem:

$$\varepsilon y'' + y = 0; \quad y(0) = 0,\ y(1) = 1 \tag{3}$$

where, for now, ε is a constant.

Solution: Identify that the linear ODE has constant coefficients. Next, write the characteristic eq.:

$$r^2 + \frac{1}{\varepsilon} = 0 \tag{4}$$

Roots: $r = \pm \frac{i}{\sqrt{\varepsilon}}$. Therefore, $y(x) = c_1 e^{\frac{i}{\sqrt{\varepsilon}}x} + c_2 e^{\frac{-i}{\sqrt{\varepsilon}}x}$.
Then, apply boundary conditions:
$y(0) = 0 \implies c_1 = -c_2 = c, \;\; y(1) = 1 \implies c = \frac{1}{2\sin(1/\sqrt{\varepsilon})}$.

Finally, $\boxed{y(x) = \dfrac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})}}$

Behavior of $y(x) = \frac{\sin(x/\sqrt{\varepsilon})}{\sin(1/\sqrt{\varepsilon})}$ as $\varepsilon \to 0^+$

Check: $\varepsilon = 1$ gives $y \sim \sin x$

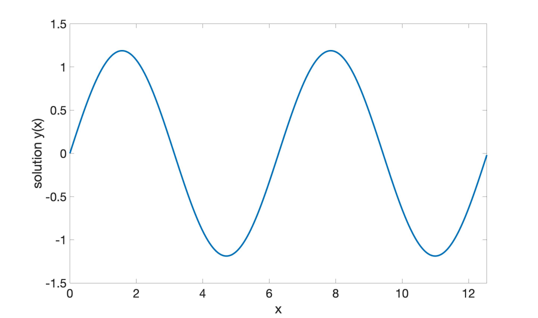

Figure: $\varepsilon = 1.0$

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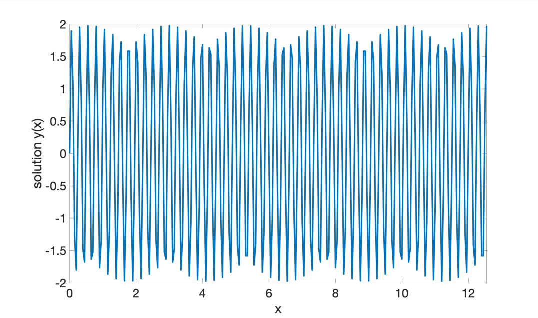

Figure: $\varepsilon = 0.0001$

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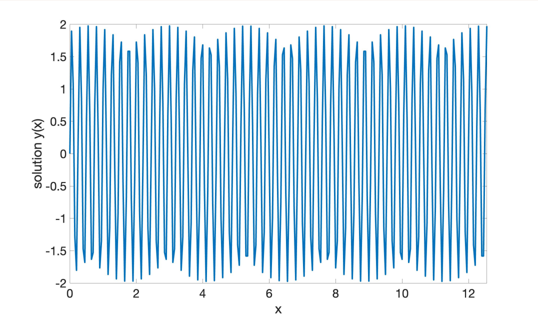

Figure: $\varepsilon = 0.0001$

For $\varepsilon = 0.0001$, we get rapid oscillations,
i.e. $y(x)$ exhibits discontinuity along x.

Singular Perturbation Problems: (a prelude to advanced mathematics for higher semesters)

Some options:

1. Since $\varepsilon \to 0^+$, ignore terms comprising ε? Thus, the ODE $\varepsilon y'' + y = 0$ becomes $y = 0$. Clearly, y(1) = 1 contradicts y(x) = 0. **BAD OPTION!**
2. WKB analysis: Seek solutions of the form

Math technique used in quantum mechanics

$$y(x) \sim e^{\frac{1}{\delta}\sum_{n=0}^{\infty}\delta^n S_n(x)}, \quad \delta \to 0 \tag{5}$$

Using (5) in $\varepsilon y'' + y = 0$, we obtain a hierarchy of closed differential equations for $S_n(x)$, solvable at every order of ε, to construct the asymptotic solution $y(x)$.