19/7/2018

pg ①

Introductory Lecture

PART (I):- Basics of probability/Statistics. L1

① Defⁿ:- (Probability) It is the measure of the likelihood that an event will occur.

(Statistics) is a branch of mathematics dealing w/ the collection, organization, analysis, interpretation & presentation of data. Often, statistical models & analysis thereof depends/relies on the principles of probability.

** But always beware that statistics tells you what may likely happen but rarely why/how something happens!

(Defⁿ) (Probability Space) is a mathematical triplet :- $(\Omega, \mathbb{F}, \mathbb{P})$

Ω : sample space - the set of all possible outcomes

$\mathbb{F}$: σ-algebra - ~~the set~~ a collection of all the events (not necessarily elementary) we would like to consider

$\mathbb{P}$: the probability measure (takes on values bet'n 0 & 1.

eg. If $X = \{a, b, c, d\}$ is a set (of outcomes) then one possible σ-algebra on X is

$$\Sigma = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}\} = \text{set of events}$$

the pair (X, Σ) is called a Borel (measurable) space.

② Axioms of Probability :- (i) If E is an event, $P(E) \geq 0$ (always. infact $P(E) \in [0, 1]$)

(ii) $P(\Omega) = 1$ i.e. probability of atleast one event (that includes nothing happening) happening out of Ω is always 1.

(iii) Let $E_1, E_2, E_3, \ldots$ be any countable sequence of disjoint sets that partition the entire sample space then

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i)$$

L2

Some other useful results

i) $P(\phi) = 0$

ii) If A & B are two events in Ω,
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

iii) If A & B are independent events then
$$P(A \cap B) = P(A) \cdot P(B)$$

iv) $P(A') = 1 - P(A)$; where A' is the complement of A.

3 Product spaces

If $(\Omega_i, \mathbb{F}_i, P_i)$; $i = 1, \ldots, n$ are probability spaces, we can let $\Omega = \Omega_1 \times \cdots \times \Omega_n = \{(\omega_1, \ldots, \omega_n) \mid \omega_i \in \Omega_i\}$
$\mathbb{F} = \mathbb{F}_1 \times \cdots \times \mathbb{F}_n$ = the σ-algebra generated by $\{A_1 \times \cdots \times A_n \mid A_i \in \mathbb{F}_i\}$. Let $P = P_1 \times \cdots \times P_n$ = the measure on $\mathbb{F}$ that has

$$P(A_1 \times \cdots \times A_n) = P_1(A_1) P_2(A_2) \cdots P_n(A_n) ?$$
$$= \frac{\text{size of } (A_1 \times \cdots \times A_n)}{\text{size of } \Omega}$$

eg: Roll 2 dice

$$\Omega = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$
$$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (2,2), (2,3) \ldots \ldots (6,1)\ (6,2) \ldots - (6,5), (6,6)\}$$

P(obtaining sum of outcomes ≥ 10)
$= \frac{6}{36} = \frac{1}{6}$

Here $A_1 = 4, 5, 5, 6, 6, 6$
$A_2 = 6, 5, 6, 4, 5, 6$

Events $(A_1 \times A_2)$
$= \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}$

$= P(A_1 \times A_2) = \frac{\text{size of } (A_1 \times A_2)}{36}$

④ Random Variable (RV) (L2)

Recall, Ω = set of outcomes.

Defn :- a RV, X is a variable whose possible values are outcomes of a random phenomenon.

$X \in \mathbb{F}$.

Technically, X defined on Ω is a RV if for every Borel set $B \subset \mathbb{R}$ we have

$$X^{-1}(B) = \{\omega \mid X(\omega) \in B\} \in \mathbb{F}$$

* If Ω is a discrete probability space then any f^n $X : \Omega \to \mathbb{R}$ is a RV.

eg. of a RV which is very very useful is the indicator f^n of a set $A \in \mathbb{F}$

$$\mathbb{1}_A(\omega) = I\{\omega \in A\} = \begin{cases} 1; & \omega \in A \\ 0; & \omega \notin A \end{cases}$$

It is also (sometimes) referred to as the characteristic f^n. of A.

⑤ Distribution of a RV

Distribution f^n / Cumulative D^n f^n.

$F(x) = P(X \leq x)$

(i) F is non-decreasing.

(ii) $F(x) \to 1$ w/ $x \to \infty$ & $F(x) \to 0$ w/ $x \to -\infty$

(iii) right continuous i.e. $\lim_{y \downarrow x} F(y) = F(x)$

(iv) If $F(x^-) = \lim_{y \uparrow x} F(y) \Rightarrow F(x^-) = P(X < x)$

L2

(v) $P(X = x) = F(x) - F(x^-)$

** for continuous R.V. $P(X = x) = 0$.

<u>Th^m</u> :- If F satisfies (i), (ii) & (iii); then F is the D^n f^n or (cdf) of some R V.

<u>Notation</u>

$$F_X(x) = P(X \le x)$$

(6) <u>Density f^n / Pdf.</u>

$$f_X(x) = \frac{d}{dx} F_X(x)$$

i.e. $F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$

Note $\int_{\Omega} f_X(x)\,dx = 1$ (Normalization; cmp. $\sum_{i \in \Omega} p_i = 1$)

<u>for discrete probability space.</u>

$f_X(x) = P(X = x)$

$F_X(x) = P(X \le x) = \sum_{x_i \le x} P(X = x_i) = \sum_{x_i \le x} f_X(x_i)$

** REFER TO THE TABULAR SHEET ENLISTING DIFFERENT TYPES OF DISCRETE & CONTINUOUS PROBABILITY DISTRIBUTION F^N_S.

L2

7 Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$ where A & B are events in $\mathcal{F}$.

eg A = event that a HT appears on 2 successive tosses of a fair coin

B = 1st toss = H

$P(A|B) = P(A \cap B)/P(B) = P(A)/P(B) = \frac{1/2 \times 1/2}{1/2} = 1/2$

This should be intuitively true b/c $P(A|B) \equiv P(2^{nd} \text{ toss} = T) = 1/2$

Pg 3

If A & B are independent events then $P(A \cap B) = P(A)P(B)$ and $P(A|B) = P(A)$ (this should intuitively make sense).

** Law of total probability :-

$$P(A) = \sum_{i=1}^{N} P(A|B_i)P(B_i)$$; here B_is partition the prob. space (into disjoint regions)

(7.1) Bayes' theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

this (should) follows immediately from the definition of ~~the~~ conditional probability stated above.

eg

Q) The entire o/p of a factory is produced on 3 m/c. The 3 m/cs account for 20%, 30% & 50% of the factory o/p. The fraction of defective items produced is 5% for the 1st m/c; 3% for the 2nd m/c; and 1% for the 3rd m/c. If an item is chosen at random from the total o/p & is found to be defective; what is the probability that it was produced by the 3rd m/c?

Ans) A_i := event that a randomly chosen item is made by i^{th} m/c ($i = 1, 2, 3$)

B := even where a randomly chosen item is defective.

$P(A_1) = 0.2$, $P(A_2) = 0.3$, and $P(A_3) = 0.5$

$P(B|A_1) = 0.05$, $P(B|A_2) = 0.03$ and $P(B|A_3) = 0.01$

$P(A_3|B) = ?$

$$P(A_3|B) = \frac{P(B|A_3)P(A_3)}{P(B)}$$

By using the law of total probability :-

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)$$

$$= 0.05 \times 0.2 + 0.03 \times 0.3 + 0.01 \times 0.5$$

$$= 0.024$$

L2

$$\therefore P(A_3|B) = \frac{P(B|A_3)P(A_3)}{P(B)}$$

$$= \frac{(0.01)(0.5)}{0.024} = 5/24$$

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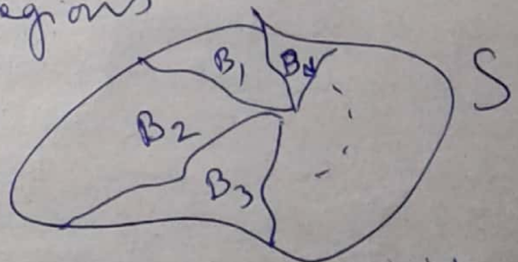
(8) Laws of total probability/expectation/variance

Let B_is partition the sample space into disjoint regions



(I) Law of total probability.

$$P(A) = \sum_{i=1}^{N} P(A|B_i)P(B_i)$$

(II) Law of total expectation

$$E(A) = \sum_{i=1}^{N} E(A|B_i)P(B_i)$$

eg $C :=$ event that 2^{nd} toss $= T$

$$P(C) = P(C|1^{st}\text{ toss}=H)P(1^{st}\text{ toss}=H) + P(C|1^{st}\text{ toss}=T) \times P(1^{st}\text{ toss}=T)$$

$$= 1/2 \times 1/2 + 1/2 \times 1/2 = 1/2 \text{ (intuitively)}$$

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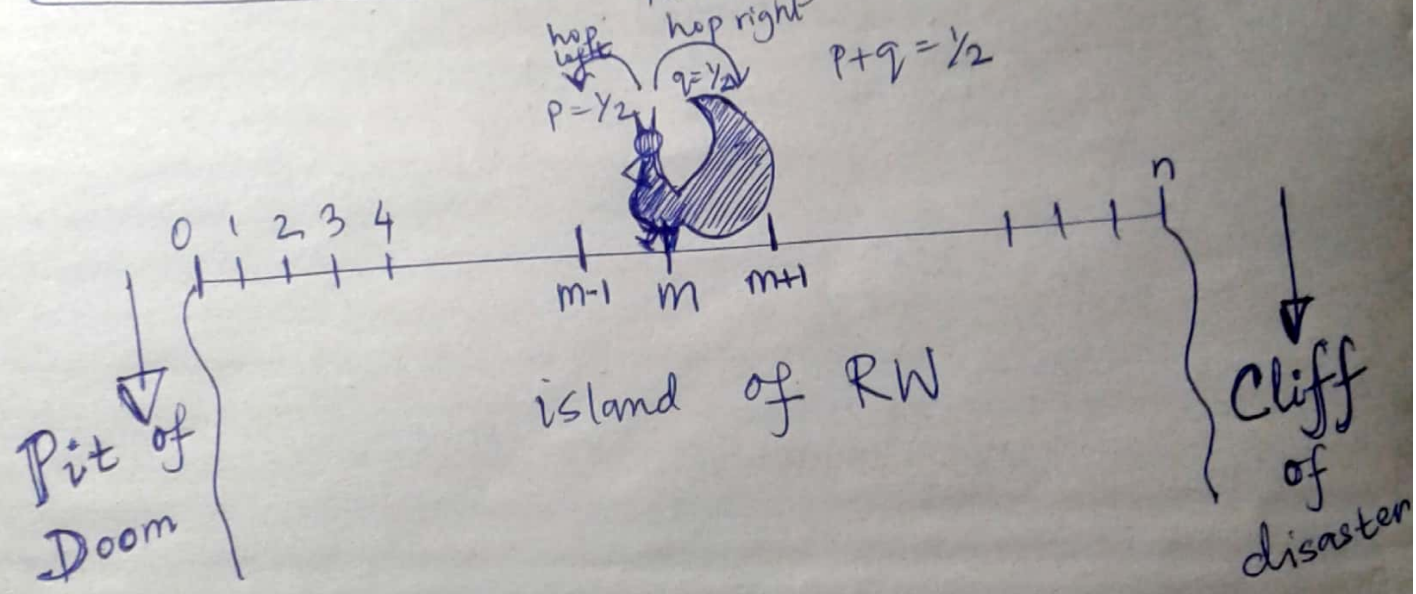
(III) Law of total variance. (L2)

$$Var(Y) = E\left[Var(Y|X)\right] + Var\left[E(Y|X)\right]$$

9) Applications of laws of total probability/expectation

Random Walk (Closely related to (Einstein's) Brownian motion).



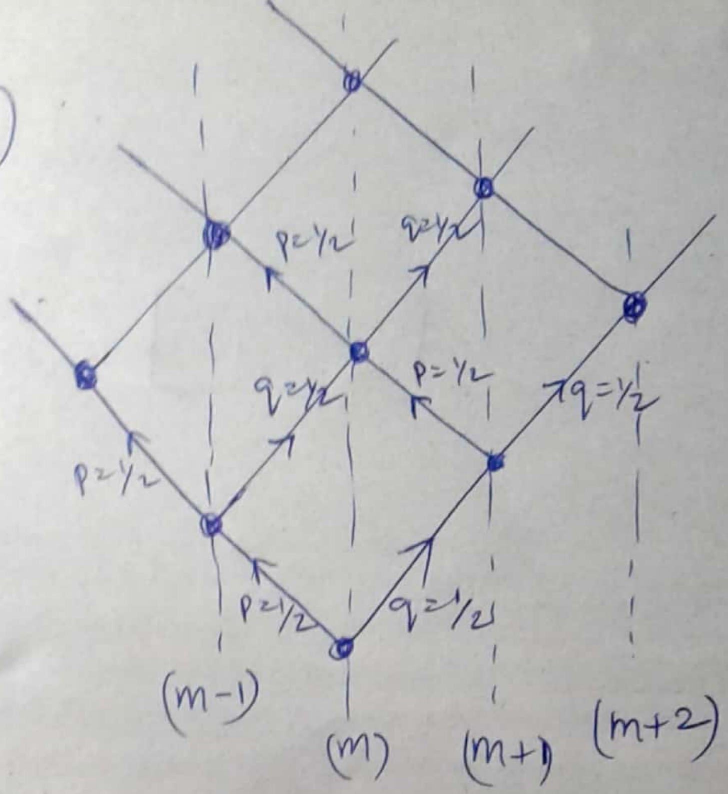
Q) (i) What is the probability that the squirrel will eventually fall off the cliff or into the pit?

(ii) What is the squirrel's life expectancy in terms of no. of hops?
Does his initial position change his chances of surviving longer?

Soln :-

(Guess)

(1)



L2

Since the tree spans the entire state space (including the boys); perhaps there is no escape for the squirrel!

W = event that the squirrel falls into the pit (left)

We want to compute P_m:

$P_m = P_m(\text{left}_{\text{pit}})$ = Prob. of W when he starts at $X_0 = m$

w/ $P_0 = 1$, $P_n = 0$

Let E be the event that the 1st hop is to the left. We will "condition" our computation upon this event E as follows

$$P_m = P(W \text{ and } E \mid X_0 = m) + P(W \text{ and } E' \mid X_0 = m)$$

Law of total probability

$$= P(W \mid E \wedge X_0 = m) P(E \mid X_0 = m) + P(W \mid E' \wedge X_0 = m) \times P(E' \mid X_0 = m)$$

$$= P(W \mid X_1 = m-1) \times \frac{1}{2} + P(W \mid X_1 = m+1) \times \frac{1}{2}$$

independent hops

$$= \frac{1}{2} P(W \mid X_0 = m-1) + \frac{1}{2} P(W \mid X_0 = m+1)$$

$$\boxed{P_m = \frac{1}{2} P_{m-1} + \frac{1}{2} P_{m+1}}$$

L3

i.e. $P_{m+1} = 2P_m - P_{m-1}$ is the recurrence relation.

Ch. eqn. $r^2 - 2r + 1 = 0 \Rightarrow r = 1, 1$ (double root)

$\therefore P_m = a \cdot 1^m + b\, m\, 1^m$ w/ $P_0 = 1, P_n = 0$

gives $P_m = 1 - \frac{m}{n}$ —— ①

Symmetrical solution

What is the probability that starting from the same initial position, he falls off the cliff on the right at $x = n$

i.e. $P_m(\text{right cliff}) = ?$

Symmetry implies $P_m(\text{right cliff})$

$= P_{n-m}(\text{left cliff})$

$= 1 - \frac{n-m}{n}$ —— ②

Now ① & ② $\Rightarrow$ $P_m(\text{left pit}) + P_m(\text{right cliff})$

$= 1 - \frac{m}{n} + 1 - \frac{n-m}{n}$

$= 2 - \frac{n}{n} - \frac{m}{n} + \frac{m}{n}$

$= 1$

i.e. the squirrel will eventually fall off the edge & die!

(iv) Let D be the no. of hops (steps) before he falls off the edge. (L3)

We will use the law of total expectation and once again condition upon the event E as follows:-

$\bar{E} = E' = E^c$ (complement of E)

$$E_m = E[D \mid X_0 = m]$$
$$= E[D \mid E \wedge X_0 = m] P(E \mid X_0 = m) + E[D \mid \bar{E} \wedge X_0 = m] P(\bar{E} \mid X_0 = m)$$
$$= \frac{1}{2} E(D \mid X_1 = m-1) + \frac{1}{2} E(D \mid X_1 = m+1)$$
$$\underset{\text{chain}}{\overset{\text{reset}}{=}} \frac{1}{2}\{1 + E(D \mid X_0 = m-1)\} + \frac{1}{2}\{1 + E(D \mid X_0 = m+1)\}$$

$$\boxed{E_m = 1 + \frac{1}{2} E_{m-1} + \frac{1}{2} E_{m+1}} \quad \text{— (3)}$$

Eqn (3) is once again a recurrence relation which can be re-written as $E_{m+1} - 2E_m + E_{m-1} = -2$ whose characteristic eqn. is $r^2 - 2r + 1 = 0$ for homogen. part with $E_0 = E_n = 0$ (by condn).

$$E_{m_{hom}} = a + bm$$

The r.h.s. is a constant which is ~~x times~~ m^0 times a in $E_{m_{hom}}$ & $(r^2 - 2r + 1) = 0$ has double root $(r = 1) \Rightarrow E_{m_{part}} = \alpha m^{0+2} (a \cancel{\ldots}) +$ all its lin. indep. derivatives

$$= a\alpha m^2 + \cancel{\ldots}^3 + \beta m + r$$
$$= Am^2 + \cancel{\ldots} + cm + r \quad \text{—(4)}$$

Let us check what values of A, c & r enable $E_{m_{part}}$ to solve (3)

$$A(m+1)^2 + c(m+1) + r - 2\{Am^2 + cm + r\} + A(m-1)^2 + c(m-1) + r = -2$$

This gives $A = -1 \Rightarrow E_{m_{part}} = -m^2$

L3

$\therefore E_m = E_{m_{hom}} + E_{m_{part}}$

$= a + bm - m^2$

Now apply bdy condn $E_0 = E_n = 0$.

$E_0 = a = 0$

$E_n = bn - n^2 = 0$

$\Rightarrow n(b-n) = 0$

$b = n$

$\therefore E_m = nm - m^2$

$$\boxed{E_m = m(n-m)}$$

i.e. the squirrel's life expectancy is the product of it's distances from the 2 edges.

** Where should the squirrel start ($x_0 = ?$) in order to have a larger life span?

Consider the f^n $f(m) = m(n-m)$

$f'(m) = n - 2m = 0$

$\Rightarrow \boxed{m = n/2}$

$f''(m) = -2 < 0$

$\therefore m = n/2$ is the locn. of maxm of $f(m)$.

Careful w/ Calculus machinery when working w/ discrete space.

$\therefore x_0 = \frac{n}{2}$ to maximize its life expectancy!

(10) Some examples of Discrete Probability Distributions

Let us first enlist some of them (L3)

i) Bernoulli D^n
ii) Geometric D^n (2 types)
iii) Binomial D^n
iv) Poisson D^n
v) (Disc) Uniform D^n
vi) Negative Binomial D^n

10.i) Bernoulli D^n

This is akin to H-T / success-failure / 0-1 type of (random) phenomenon.

$X \sim \text{Bernoulli}(p)$

$$f_X(x) = \begin{cases} p \; ; & x = 1 \\ (1-p) \; ; & x = 0 \end{cases}$$

$E(X) = p$

$Var(X) = p(1-p)$

$$Var(X) = E(X^2) - (E(X))^2$$
$$= 1^2 \cdot P(X=1) + 0^2 P(X=0) - p^2$$
$$= p - p^2$$
$$= p(1-p)$$

eg. toss a fair coin
Heads w/ $p = 1/2$
Tails w/ $p = 1/2$
X is the RV that denotes the outcome of a single toss.
$E(X) = p = 1/2$
i.e. $1/2$ the times we toss the coin we will see heads (& $1/2$ the times tail)

10.ii) Geometric D^n of type 1

X = # trials until 1st success

$X \sim \text{geom}_1(p)$

$$f_X(x) = \begin{cases} (1-p)^{x-1} \cdot p \; ; & x = 1, 2, 3 \ldots \\ 0 \; ; & \text{o.w} \end{cases}$$

$E(X) = \frac{1}{p}$; $Var(X) = \frac{(1-p)}{p^2}$

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L4

$$E(X) = \sum_{x=1,2\ldots} x P(X=x)$$
$$= \sum_x x(1-p)^{x-1} p = p \sum_x x(1-p)^{x-1}$$
$$= p\left\{1 + 2(1-p) + 3(1-p)^2 + 4(1-p)^3 + \cdots\right\}$$

Let $S = \sum x(1-p)^{x-1}$

$$S_1 = \sum_x (1-p)^x = \frac{1}{1-(1-p)} = \frac{1}{p}$$

$$\frac{\partial S_1}{\partial p} = \underbrace{\sum_x x(1-p)^{x-1}(-1)}\, = -\frac{1}{p^2}$$

$$\Rightarrow \sum_x x(1-p)^{x-1} = \frac{1}{p^2}$$

$$\therefore S = \frac{1}{p^2}$$

and hence $E(X) = \frac{1}{p}$

$$Var(X) = E(X^2) - (E(X))^2$$

$E(X^2) = ?$

$$E(X^2) = E(X(X-1)) + E(X)$$ (Note Exp. operator is linear).

$$E[X(X-1)] = \sum_x x(x-1)(1-p)^{x-1}p$$
$$= p\sum_x (x-1)\, x(1-p)^{x-1}$$
$$= (-1)p \sum_x \left\{\frac{d}{dp}(1-p)^x\right\}(x-1); \quad \text{b/c } \frac{d}{dp}(1-p)^x = x(1-p)^{x-1}(-1)$$
$$= -p\sum_x (x-1)\frac{d}{dp}(1-p)^x$$
$$= -p\frac{d}{dp}\sum_{x=1,\ldots}(x-1)(1-p)^x$$
$$= -p\frac{d}{dp}\left[(1-p)^2 \sum_{x=1,\ldots}(x-1)(1-p)^{x-2}\right]$$
$$= -p\frac{d}{dp}\left[(1-p)^2\sum_{x=2}(x-1)(1-p)^{x-2}\right]$$

L4

$$= -p\frac{d}{dp}\left[(1-p)^2 \sum_{x=1,\dots} x(1-p)^{x-1}\right]$$

$$= -p\frac{d}{dp}\left[(1-p)^2(-1)\frac{d}{dp}\sum_{x=1,\dots}(1-p)^x\right]$$

$$= p\frac{d}{dp}\left[(1-p)^2\frac{d}{dp}\sum_{x=1,\dots}(1-p)^x\right]$$

$$= p\frac{d}{dp}\left[(1-p)^2\frac{d}{dp}\left(\frac{1}{p}\right)\right]$$

$$= p\frac{d}{dp}\left((1-p)^2\left(-\frac{1}{p^2}\right)\right)$$

$$= -p\frac{d}{dp}\left[\frac{1}{p^2} - \frac{2}{p} + 1\right]$$

$$= -p\left\{(-2)\frac{1}{p^3} + \frac{2}{p^2}\right\}$$

$$= \frac{2}{p^2} - \frac{2}{p} = 2\frac{(1-p)}{p^2}$$

$$\therefore \ Var(x) = E(x^2) - (E(x))^2$$

$$= \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{2-2p+p-1}{p^2} = \frac{1-p}{p^2}$$

#.

Geom D^n of type (0)

Y = # failures before 1^{st} success

$$f_Y(y) = \begin{cases}(1-p)^y p & ; y = 0,1,2\dots \\ 0 & ; o.w.\end{cases}$$

$$Y \sim geom_0(p)$$

$$E(Y) = \frac{1-p}{p}$$

$$Var(Y) = \frac{1-p}{p^2}$$

#

10.iii) Binomial D^n

Z = # successes in n independent trials w/ success probability p.

i.e. $Z \sim \text{Binomial}(n, p)$

L4

$$f_Z(z) = \begin{cases} {}^nC_z \, p^z (1-p)^{n-z} ; & z = 0, 1, 2, \ldots n \end{cases}$$

$$E(Z) = np$$
$$Var(Z) = np(1-p)$$

** $Z_1 \sim Bin(n, p)$
$Z_2 \sim Bin(m, p)$
$\Rightarrow Z_1 + Z_2 \sim Bin(n+m, p)$.

10.iv) Poisson D^n

(eg. No. of arrivals in a given time interval).

$X \sim \text{poisson}(\lambda)$; $\lambda \in \mathbb{R}$ (rate of arrivals)

$$f_X(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

$$E(X) = \lambda$$
$$Var(X) = \lambda$$

Note :- Both are same!

We will discuss poisson processes in more detail later in the course.

** Poisson D^n is a good approx^n for $Bin(n, p)$ for n large & small p w/ $\lambda = np$.

10.v) Discrete Uniform D^n

$X \sim \text{Unif}[a, b]$

$n = b - a + 1$

$f_X(x) = \frac{1}{n}$; $E(X) = \frac{a+b}{2}$; $Var(X) = \frac{(b-a+1)^2 - 1}{12}$

10-VI) Negative Binomial D^n.

There are other well Known D^n like Pascal D^n & Polya D^n that are special cases of the -ve Binomial D^n. (L4)

X = # successes in a sequence of independent and identically distributed (iid) Bernoulli trials before a specified no. of failures (say r) occurs.

$$f_X(x) = {}^{x+r-1}C_x (1-p)^r p^x$$

$$E(X) = \frac{pr}{1-p}$$

$$Var(X) = \frac{pr}{(1-p)^2}$$

Some solved examples & applications of Discrete probability D^n.

(A) geometric D^n.

Q1) If your probability of success of meeting a Congress voter is 0.2; what is the prob. you meet a congress voter on your 3^{rd} ~~try~~ (meet)?

Ans :- $p = 0.2$

X = # trials ~~before~~ until 1^{st} success (including the successful meet)

$X \sim geom_1(p)$

$f_X(x) = (1-p)^{x-1} p \; ; \; x = 1, 2, \ldots$

3^{rd} try means

$$P(X=3) = f_X(3) = (1-p)^{3-1} p = (0.8)^2 \, 0.2 = 0.128$$

(B) Binomial D^n

Q1) A binary source generates digits 1 and 0 randomly w/ probabilities 0.6 and 0.4, respectively

(i) What is the probability that two 1s and three 0s will occur in a 5-digit sequence?

(ii) What is the probability that atleast 3 1s will occur in a 5-digit sequence?

Soln :- (i) $X = \#\, 1s$ in a 5 digit seq.

$$X \sim Bin(5, 0.6)$$

$$P(X=2) = {}^5C_2 (0.6)^2 (0.4)^3 = 0.23$$

(ii) $P(X \geq 3) = 1 - P(X \leq 2)$

$$= 1 - \left\{ \sum_{k=0}^{2} {}^5C_k (0.6)^k (0.4)^{5-k} \right\}$$

$$= 1 - 0.317 = 0.683$$

b/c these are disjoint events

(C) Poisson D^n

Q1) The no. of telephone calls arriving at a switchboard during any 10 minute period is known to be a poisson RV X w/ $\lambda = 2$ (rate)

(i) Find the probability that more than 3 calls will arrive during any 10-minute period.

(ii) Find the probability that no calls will arrive during any 10 minute period.

Soln:-

(i) $P(X > 3) = 1 - P(X \le 3)$

$$= 1 - \sum_{k=0}^{3} \frac{e^{-2} 2^k}{k!}$$

$$= 1 - e^{-2}\left(1 + 2 + \frac{4}{2} + \frac{8}{6}\right) \approx 0.143$$

L5

(ii) $P(X = 0) = e^{-2} \approx 0.135$

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11 Some examples of Continuous probability D^n

11.1) Normal D^n

$X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad ; \ \sigma > 0$$

$E(X) = \mu, \ Var(X) = \sigma^2$

11.2) Uniform D^n

$X \sim unif(a, b)$

$$f_X(x) = \frac{1}{b-a} \quad ; \ a \le x \le b$$

$$E(X) = \frac{b+a}{2} \ ; \quad var(X) = \frac{(b-a)^2}{12}$$

11.3) Pareto D^n (Distribution of wealth in a society — fitting the trend that a large portion of wealth is held by a small fr. of the population)

$X \sim Pareto(\alpha, \beta)$

scale (α), shape (β)

$$f_X(x) = \frac{\beta \alpha^\beta}{x^{\beta+1}} \ ; \ x > \alpha ; \ \alpha, \beta > 0$$

$$E(X) = \frac{\beta\alpha}{\beta - 1} \ ; \ \beta > 1 \ (\infty \text{ o.w.})$$

$$Var(X) = \frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)} \ ; \ \beta > 2 \ (\infty \text{ o.w.}) \ (\text{mgf DNE?}).$$

11.4) Gamma D^n (Waiting time D^n) L5

* Exponential, χ^2 are spl. cases of Γ D^n.

$X \sim \Gamma(\alpha, \beta)$ — α → shape, β → scale

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$$

$E(X) = \alpha\beta$

$var(X) = \alpha\beta^2$

11.5) Exponential D^n (Waiting times)

$X \sim exp(\theta)$; $\theta \sim \frac{1}{\lambda}$ where λ is rate (Poisson)

$f_X(x) = \frac{1}{\theta} e^{-x/\theta}$; $x \geq 0$; $\theta > 0$

$E(X) = \theta$; $var(X) = \theta^2$

11.6) χ^2, t_ν, F_{ν_1, ν_2} are sampling D^ns & we will study them later.

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(12) Joint D^ns & Marginal D^n

We will consider only the bivariate case

X, Y are RV

eg: 2 independent coins are flipped each w/ success (Heads) probability $= 1/2$

$P(A) = 1/2$ for $A \in \{0, 1\} \equiv \{T, H\}$

$P(B) = 1/2$ for $A \in \{0, 1\} = \{T, H\}$

Sample space of jt. D^n $(A, B) = \{(0,0), (0,1), (1,0), (1,1)\}$