Matrix and vector norms

Let $\vec{x} \in \mathbb{R}^n$ i.e. $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

A vector norm on $\mathbb{R}^n$ is a f^n $\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$ w/ the following properties

(1) $\|\vec{x}\| \geq 0 \quad \forall \vec{x}$

(2) $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$

(3) $\|\alpha \vec{x}\| = |\alpha| \|\vec{x}\| \quad \forall \alpha \in \mathbb{R}, \forall x \in \mathbb{R}^n$

(4) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\| \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^n$

There are <u>many types</u> of norms

(1) $\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

(2) $\|\vec{x}\|_\infty = \max_{1 \leq i \leq n} |x_i|$

* All norms are equivalent!

Cauchy-Schwarz inequality

$$\left|\langle \vec{x}, \vec{y} \rangle\right| \equiv \left|\vec{x}^{T}\vec{y}\right| = \left|\sum_{i=1}^{n} x_i y_i\right| \le \|\vec{x}\|_2 \|\vec{y}\|_2$$

What is the use of norms?

(1) To measure distance b/n 2 pts. in space.

(2) Convergence of sequences (eg. analysis/ monitoring of error of iterative methods)

A sequence $\{\vec{x}_{(k)}\}_{k=1}^{\infty}$ of vectors in $\mathbb{R}^n$ is said to converge to $\vec{x}$ w.r.t. the norm $\|\cdot\|$ if for any small $\epsilon > 0$, $\exists\ N(\epsilon)$ s.t. $\|\vec{x}_{(k)} - \vec{x}\| < \epsilon \quad \forall\ k \ge N(\epsilon)$

Th^m $\{\vec{x}_{(k)}\} \to \vec{x}$ in $\mathbb{R}^n$ w.r.t. $\|\cdot\|_\infty \iff \lim_{k\to\infty} x_{i(k)} = x_i$ for each $i = 1, 2, \ldots, n$

Th^m for each $\vec{x} \in \mathbb{R}^n$

$$\|\vec{x}\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|\vec{x}\|_\infty$$

X

Matrix norms

Let $A \in M_{n\times n}$; $\|\cdot\|$ is a f^n that maps A in $M_{n\times n}$ to a real-valued no.

properties of matrix norms

(1) $\|A\| \geq 0$

(2) $\|A\| = 0 \iff A$ is a 'O' matrix w/ all entries $= 0$.

(3) $\|\alpha A\| = |\alpha| \|A\|$

(4) $\|A+B\| \leq \|A\| + \|B\|$

(5) $\|AB\| \leq \|A\| \|B\|$

thm (Induced matrix norm)

If $\|\cdot\|$ is a vector norm on $\mathbb{R}^n$ then

$$\|A\| := \max_{\|\vec{x}\|=1} \|A\vec{x}\| \text{ is a matrix norm.}$$

$$\max_{\vec{z}\neq\vec{0}} \left\| A\left(\frac{\vec{z}}{\|\vec{z}\|}\right)\right\| = \max_{\vec{z}\neq 0} \frac{\|A\vec{z}\|}{\|\vec{z}\|}$$

Th^m^ ($\|\cdot\|_\infty$ matrix norm).

def $A = (a_{ij}) \in M_{n\times n}$;

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$$

eg. $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 5 & -1 & 1 \end{pmatrix}$

$\longrightarrow \sum_{j=1}^{3} |a_{1j}| = |1| + |2| + |-1| = 4$

$\longrightarrow \sum_{j=1}^{3} |a_{2j}| = 4$

$\longrightarrow \sum_{j=1}^{3} |a_{3j}| = 7$

$\therefore \|A\|_\infty = \max\{4, 4, 7\} = 7$

Spectral radius of a matrix

$\rho(A) := \max_{1 \le i \le n} |\lambda_i|$; where $\lambda_1, \ldots, \lambda_n$ are evs of $A \in M_{n \times n}$

* spectral radius is closely related to the norm of a matrix !

Th^m :- Let $A \in M_{n \times n}$;

(i) $\|A\|_2 = \sqrt{\rho(A^+ A)}$

(ii) $\rho(A) \le \|A\|$ where $\|\cdot\|$ is an induced matrix norm.

A simple matrix decomposition

Consider any matrix, say

$$A = \begin{pmatrix} 2 & -1 & 5 \\ 0 & 1 & -2 \\ 1 & 5 & -1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 5 & 0 \end{pmatrix}} + \underbrace{\begin{pmatrix} 0 & -1 & 5 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}}$$

Can always do this splitting!

$$= D + L + U$$

(strictly lower triangular) (strictly upper triangular)

$$= D - (-L) - (-U)$$

Iterative Schemes to solve systems of linear eqns.

The idea is to solve $A\vec{x} = \vec{b}$ by rewriting it in the form $\vec{x} = T\vec{x} + \vec{c}$ & then using an iterative scheme of the form $\vec{x}_{(k)} = T\vec{x}_{(k-1)} + \vec{c}$ where $k = 1, 2, 3 \cdots$

Jacobi iterative method

eg. Let us consider the system of linear eqns

$$E_1: \quad 10x_1 - x_2 + 2x_3 = 6$$
$$E_2: \quad -x_1 + 11x_2 - x_3 + 3x_4 = 25$$
$$E_3: \quad 2x_1 - x_2 + 10x_3 - x_4 = -11$$
$$E_4: \quad 3x_2 - x_3 + 8x_4 = 15$$

which has a unique soln $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$.

Consider this in the form $A\vec{x} = \vec{b}$ where

$$A = \begin{pmatrix} 10 & -1 & 2 & 0 \\ -1 & 11 & -1 & 3 \\ 2 & -1 & 10 & -1 \\ 0 & 3 & -1 & 8 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 6 \\ 25 \\ -11 \\ 15 \end{pmatrix}$$

Let us re-write $A\vec{x}=b$ as $\vec{x}=T\vec{x}+\vec{c}$ thusly (4)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \begin{aligned} &= \quad \tfrac{1}{10}x_2 - \tfrac{1}{5}x_3 + \tfrac{3}{5} \\ &= \tfrac{1}{11}x_1 + \tfrac{1}{11}x_3 - \tfrac{3}{11}x_4 + \tfrac{25}{11} \\ &= -\tfrac{1}{5}x_1 + \tfrac{1}{10}x_2 + \tfrac{1}{10}x_4 - \tfrac{11}{10} \\ &= \quad -\tfrac{3}{8}x_2 + \tfrac{1}{8}x_3 + \tfrac{15}{8} \end{aligned}$$

These unknowns were already computed in the eqns above.

$$\Rightarrow \vec{x} = \begin{pmatrix} 0 & \frac{1}{10} & -\frac{1}{5} & 0 \\ \frac{1}{11} & 0 & \frac{1}{11} & -\frac{3}{11} \\ -\frac{1}{5} & \frac{1}{10} & 0 & \frac{1}{10} \\ 0 & -\frac{3}{8} & \frac{1}{8} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 3/5 \\ 25/11 \\ -11/10 \\ 15/8 \end{pmatrix}$$

$$\vec{x} = T\vec{x} + \vec{c}$$

initial guess (say) $\vec{x}_{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ then $\vec{x}_{(1)} = \begin{pmatrix} 0.6 \\ 2.2727 \\ -1.1000 \\ 1.8750 \end{pmatrix}$

Carry forth the computations iteratively

$$\cdots \quad \vec{x}_{(10)} = \begin{pmatrix} 1.0001 \\ 1.9998 \\ -0.9998 \\ 0.9998 \end{pmatrix}$$

Whence $$\frac{\|\vec{x}_{(10)} - \vec{x}_{(9)}\|}{\|\vec{x}_{(10)}\|} < 10^{-3}$$

STOP!

Jacobi iteration (in matrix form)

$$A\vec{x} = b$$

$$\Rightarrow (D+L+U)\vec{x} = b$$

$$\Rightarrow D\vec{x} = -(L+U)\vec{x} + \vec{b}$$

$$\Rightarrow \vec{x} = -D^{-1}(L+U)\vec{x} + \vec{b}$$

TYPO: There is a inv(D) missing here!

Hence the iteration

$$\vec{x}_{(k)} = -D^{-1}(L+U)\vec{x}_{(k-1)} + \vec{b}$$

in the form of iterates we have

$$x_{i(k)} = \frac{\sum_{\substack{j=1\\ j\neq i}}^{n}\left(-a_{ij}\,x_{j(k-1)}\right) + b_i}{a_{ii}} \; ; \; i = 1, 2, \ldots, n$$

$a_{ii} \neq 0 \Rightarrow D$ is invertible!

Gauss - Seidel iterative scheme

$$x_{i(k)} = \frac{-\sum_{j=1}^{i-1} a_{ij} x_{j(k)} - \sum_{j=i+1}^{n} a_{ij} x_{j(k-1)} + b_i}{a_{ii}} \; ; \quad i = 1, 2, \ldots n$$

Why is this a good idea?

Compare w.r.t. the example at the beginning of this lecture!

in matrix form

$$A\vec{x} = \vec{b}$$

$$(D+L)\vec{x} = -U\vec{x} + \vec{b}$$

as iterates $(D+L)\vec{x}_{(k)} = -U\vec{x}_{(k-1)} + \vec{b}$

or $\vec{x}_{(k)} = -(D+L)^{-1}U\vec{x}_{(k-1)} + (D+L)^{-1}\vec{b}$; $k=1,2,\ldots$

Gauss-Seidel iteration

*In Next lecture, we will talk about Convergence of iterative schemes & also SOR scheme!