Question 1

(1) (i)

Given that

$$\langle v, w \rangle = |v|\,|w| \cos\theta$$

Now, we want to calculate $\langle v_1, v_2 \rangle$ when $v_1 \perp v_2$

so the angle between v_1 and v_2 is 90°

hence
$$\langle v_1, v_2 \rangle = |v_1|\,|v_2| \cos 90^\circ$$
$$= |v_1|\,|v_2| \times 0 = 0$$

hence $\boxed{\langle v_1, v_2 \rangle = 0}$

(ii)

$$\begin{array}{c} A = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 2 & 6 \end{pmatrix} \\ \quad\;\; \downarrow \;\; \downarrow \;\; \downarrow \\ \quad\;\; v_1 \;\; v_2 \;\; v_3 \end{array}$$

Let

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 9 \\ 3 \\ 0 \\ 6 \end{pmatrix}$$

Now

$$\langle v_1, b_1\rangle = \left\langle \begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\\2\end{pmatrix}\right\rangle$$

$$= -1 - 1 + 0 + 2 = 0$$

$$\langle v_1, b_1\rangle = 0$$

again

$$\langle v_2, b_1\rangle = \left\langle \begin{pmatrix}3\\1\\0\\2\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\\2\end{pmatrix}\right\rangle = -3 - 1 + 0 + 4 = 0$$

and

$$\langle v_3, b_1\rangle = \left\langle \begin{pmatrix}9\\3\\0\\6\end{pmatrix}, \begin{pmatrix}-1\\-1\\0\\2\end{pmatrix}\right\rangle = -9 - 3 + 0 + 12 = 0$$

Hence

$$\langle v_1, b_1\rangle = \langle v_2, b_1\rangle = \langle v_3, b_1\rangle = 0$$

so b_1 is perpendicular to all the columns of matrix A.

hence, b_1 is also perpendicular to the span of columns of A.

so b_1 does not belongs to the column space of A.

$\Rightarrow \boxed{Ax = b \text{ has no solution}}$

Solution to Question-2

(ii) Let T be the linear transformation as

$$T: \mathbb{P}_2 \longrightarrow \mathbb{P}_2 \text{ with } T(f(x)) = f'(x) + f''(x)$$

where $\mathbb{P}_2$ is space of polynomials upto degree 2.

Let $\mathcal{B}$ be basis of space $\mathbb{P}_2$ such that

$$\mathcal{B} = \{1, x, x^2\}$$

then, the matrix representation of T will be denoted as $\mathcal{B}$ whose order is 3×3. such that

$$\mathcal{B} = \begin{pmatrix} | & | & | \\ T(1) & T(x) & T(x^2) \\ | & | & | \end{pmatrix}$$

$$T(1) = 0 + 0 = 0 = 0.1 + 0.x + 0.x^2$$

$$T(x) = 1 + 0 = 1 = 1.1 + 0.x + 0.x^2$$

$$T(x^2) = 2x + 2 = 2.1 + 2.x + 0.x^2.$$

$$\mathcal{B} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

(ii) $q(x) = -x + 3 =$

$$q \equiv \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$$T(q) = Bq = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$$

$$\equiv \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$T(q) = -1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 = -1.$$

Question 3

9. **Definition (Kernel of T (or equivalently the null space of A, $Null(A)$)):** The set of all $x \in \mathbb{R}^n$ s.t. $T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$.

Q) Find a basis of the kernel of A (equivalently, $Null(A)$) and determine $dim(Ker(A)) = dim(null(A))$.

Ans) Most importantly $Ker(A) = Ker(rref(A)) = Ker(B)$. So we might as well solve for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ s.t. $B\mathbf{x} = \mathbf{0}$. This is done by considering the augmented matrix $\tilde{B} = \begin{pmatrix} B & | & \mathbf{0} \end{pmatrix}$ from which we have the following:

$$x_1 + 2x_2 + 0x_3 + 3x_4 - 4x_5 = 0$$
$$0x_1 + 0x_2 + x_3 - 4x_4 + 5x_5 = 0$$

or equivalently,

$$x_1 = -2x_2 - 3x_4 + 4x_5$$
$$x_3 = 4x_4 - 5x_5$$

whence $x_2 = \alpha, \;\; x_4 = \beta, \;\; x_5 = \gamma$ are set arbitrarily. Therefore,

$$\mathbf{x} = \begin{pmatrix} -2\alpha - 3\beta + 4\gamma \\ \alpha \\ 4\beta - 5\gamma \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -2\alpha & -3\beta & +4\gamma \\ \alpha & & \\ & 4\beta & -5\gamma \\ & \beta & \\ & & \gamma \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix}.$$

The $Null(A)$ is spanned by these basis vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix}$ and $dim(Null(A)) = 3$.

Question 4

8. **Definition (Image or range of a matrix/linear transformation):**

$Im(A) = Im(T)$ is the *span* of the column vectors of A.

Q) Find a basis of the image of $A = \begin{pmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{pmatrix} = \begin{pmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{pmatrix}$ and determine $dim(Im(A))$.

Ans) To find the basis of Im(A), we need to identify the redundant columns of A from amongst all the column vectors of A. By inspection of A, it will be hard to tell which of the columns of A are redundant (linearly dependent on the others). So we will transform A to B = rref(A).

$$B = rref(A) = \begin{pmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} | & | & | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \\ | & | & | & | & | \end{pmatrix}.$$

The redundant columns of B correspond to the redundant columns of A. The redundant columns of B are also easy to spot: *They are the columns that do not contain a leading 1*, namely, $\mathbf{b}_2 = 2\mathbf{b}_1$, $\mathbf{b}_4 = 3\mathbf{b}_1 - 4\mathbf{b}_3$, and $\mathbf{b}_5 = -4\mathbf{b}_1 + 5\mathbf{b}_3$. Thus the redundant columns of A are $\mathbf{a}_2 = 2\mathbf{a}_1$, $\mathbf{a}_4 = 3\mathbf{a}_1 - 4\mathbf{a}_3$, and $\mathbf{a}_5 = -4\mathbf{a}_1 + 5\mathbf{a}_3$. And the non-redundant columns of A are $\mathbf{a}_1$ and $\mathbf{a}_3$, they form a basis of image of A. Therefore, a basis of image of A is

$$\begin{pmatrix} 1 \\ -1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \\ 1 \end{pmatrix}$$

$dim(Im(A)) = 2.$

Question 5

$$x_1 + 4x_2 + 2x_3 = -2$$
$$-2x_1 - 8x_2 + 3x_3 = 32$$
$$x_2 + x_3 = 1$$

$$A = \begin{pmatrix} 1 & 4 & 2 \\ -2 & -8 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\Big\downarrow R_2 - (-2)R_1 \to R_2$$

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 7 \\ 0 & 1 & 1 \end{pmatrix}$$

Clearly it will not be possible to perform row-redn w/o row swapping; so let's try to solve the equivalent eqs.

$$x_1 + 4x_2 + 2x_3 = -2$$
$$x_2 + x_3 = 1$$
$$-2x_1 - 8x_2 + 3x_3 = 32$$

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ -2 & -8 & 3 \end{pmatrix}$$ * No transfn was reqd for R_2 $\therefore m_{21} = 0$

$$\Big\downarrow R_3 - (-2)R_1 \to R_3$$

$m_{31} = -2$

$$U = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{pmatrix}$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$ Indeed $A = LU$

1^{st} solve $\quad L\vec{y} = \vec{b}$ —— (I)

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 32 \end{pmatrix}$$

fwd sub $\Rightarrow$ $y_1 = -2$

$y_2 = 1$

$-2y_1 + y_3 = 32$

$\Rightarrow y_3 = 32 + 2\times(-2)$

$= 28$

then solve

$$U\vec{x} = \vec{y} \quad \text{—— (II)}$$

$$\Rightarrow \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 28 \end{pmatrix}$$

bkd sub $\Rightarrow$ $7x_3 = 28 \Rightarrow x_3 = 4$

$x_2 + x_3 = 1 \Rightarrow x_2 = -3$

$x_1 + 4x_2 + 2x_3 = -2$

$\Rightarrow x_1 + (-12) + 8 = -2$

$\Rightarrow x_1 = 2$

$$\therefore \vec{x} = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}$$

this is exactly the answer we had obtained!